MATHEMATICS MADE MEANINGFUL

contemporary implementation of the Waldorf Curriculum
for classes 1 – 7

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the need for children to:
fully understand mathematical concepts,
be free to choose their own solutions,
and practise skills thoroughly

the need for teachers to:
make the right choices in how and when to teach what
be aware of different levels, learning styles and levels of competence
avoid teaching tricks, but always work with numerical understanding

including:
The Four Processes, by Yvonne Bleach, 2003 (adapted)

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THE BIG AIM: UNDERSTANDING MATHS, INSTEAD OF TRICKS

We have, for years, become used to the unfortunate pattern that many children start off doing quite well in Maths, but somewhere down the line, already in primary school or later in high school, drop the ball. One of the main reasons for this pattern is that teaching and learning mathematics is often mainly about how to find answers, not how to understand the mathematical principles that lie behind the answers.

The most prominent example behind this statement is the set of vertical algorithms that we have been teaching our children, through which they could find answers to additions and subtractions, multiplications and divisions, by calculating digit-by-digit, following a series of instructions, and producing an answer that would hopefully be marked as correct. The children would often have hardly any sense whether their answer would be right or wrong.

In long division, for instance, the series of instructions was elaborate, and children were trained to follow each step, ie.

OLD STYLE LONG DIVISION	1761 ÷ 3 =	
	587	
	3	1761	how many times does 3 fit into 17?
	15	5 x 3 = 15
	26	subtract and drop the 6, how many times does 3 fit into 26?
	24	8 x 3 = 24
	21	subtract and drop the 1, how many times does 3 fit into 21?
	21	7 x 3 = 21
	0	done, no remainder

This system works very well, it is manageable for most children and it can tackle even complicated situations with decimal numbers and all. However, the children's focus on the digits completely removes them from the actual question. They find a 5 x 3, an 8 x 3 and a 7 x 3, but hardly relate this to the problem they are solving. Very few children end this calculation with an inner connection to the conclusion that 587 fits 3 times into 1761. They have mechanically resolved the problem, without using real mathematical thinking.

Although many children manage to learn primary maths by following these rules and instructions, and seemingly producing work that shows their "understanding", in fact these algorithms are doing the children a disservice. Instead, they could and should be practising real understanding of the numbers and of the operations they are carrying out. This means that we need to replace the vertical approach, which is often "the" way children do maths from class 2, to a radically different one, which enhances children's understanding.

In learning mathematics one heavily depends on each learning step that is taken, as all mathematical concepts in higher grades hinge upon a thorough level of understanding the previous work. This is why even children who were functioning well with the vertical approach, up to a certain level, drop out of maths in high school, when the numbers of tricks and rules to apply become so overwhelming that the children no longer manage to understand which rule applies to which situation.

In the South African National Curriculum there is a strong realisation of the need to change our approach to maths teaching. Many teachers are still strongly rooted in the old approach, and even text book writers are not always conscious of the pitfalls to avoid. However, as a policy the educational leaders of SA firmly stand for the new mathematical approach described in this booklet, and support that children should not be doing vertical algorithms before class 4.

In Waldorf schools, many teachers are very skilled in bringing mathematics in a lively way, full of movement and images, but once it comes to the written calculation skills, they also fall back upon what they have learnt themselves, not realising the shortcomings of that tradition.

Spearheaded by the late Yvonne Bleach, much work has been done on changing the teaching of mathematics to an approach that supports mathematical understanding. This booklet was written in support of Yvonne's work, with various additions resulting from teaching experiences with primary children as well as teacher training students.

EXPANDED CALCULATION, BEFORE VERTICAL APPROACH

For many years, the vertical algorithms for additions, subtractions and multiplications, and the long division, have been the main approach towards teaching children these calculations, after the level of questions that could be solved by counting or with mental maths. The problem with these algorithms is that they are happening digit by digit, see the example given in the paragraph about place value (page 19), and that the children no longer connect to the full numbers, the actual operation, and therefore not to the answer either. Children often come to an answer, digit by digit, without having a clue whether their answer is right or wrong. They have merely learnt to perform a process with the single digits, which is disconnected from the problem statement.

The old, vertical approach is one of the examples of an approach to learning mathematics that was in fact based on teaching children a set of tricks, that enabled them to come to answers that might be right, but actually did not equip them with real understanding of the mathematical work they did. Most children could sustain this level of learning for a while, and seemingly do quite well in Maths, but when the numbers of tricks they learnt became more and more, and the choices (which trick applied to which situation) more complex, one by one children dropped out of the expected level and became pupils who failed Maths. Due to their lack of basic understanding, the remediation of their mathematical skills was an impossible task.

From the above we have learnt that the aim of teaching mathematics needs to be a thorough understanding of the mathematical processes and operations. Such understanding needs to be based on working with the real numbers, and therefore must avoid the digit-by-digit approach, at least for several years of primary education. The South African National Curriculum now expects teachers to abstain from vertical algorithms until grade 4.

Before children work with vertical solutions, there are possibilities to work out sums in a horizontal way, that deals with the real numbers, and uses structured manipulation of number parts of similar value (e.g. hundreds, tens, units) to tackle sums that are too complicated for mental solutions.

After having been introduced, in class 1, to the four operations, and having practised sums that could be worked out with counters, or mentally with the help of the bonds, children from class 2 upward need to tackle more complex questions sums as 25 + 18. The term “expanded calculation” mainly refers to the technique of decomposing numbers (in this example in tens and units), as 25 = 20 + 5 and 18 = 10 + 8. Originally, this was described with the term “expanded notation”, which meant the same, but could be interpreted as merely another way of writing down one’s calculations. An unspoken connotation of “expanded notation” was that all the numbers were decomposed, often leading to endless series of numbers and operations.

Our approach “expanded calculation” includes making the right choices about decomposition. Conscious of the Three Levels of Operations (page 4), it enables us to move from expanding (decomposing) all the numbers towards only expanding the numbers we need to expand, to be able to calculate the answer.

The possibilities with the expanded calculation will now be investigated in the light of the ‘Three Levels of Operations’ (page 4). For each operation (except subtractions) the count-all, add-on and regrouping levels are shown. Generally, the introduction to a new concept is brought, by the teacher, in the count-all form. Children start operating the numbers this way, while being encouraged to find their own solutions. This will soon lead to some children presenting solutions at a higher level. As described in the paragraph ‘different strategies’ (page 21), the teacher and the class compare how various children have found their solutions, and bit by bit, the whole class should reach at least the second level (add-on) which is a more economical approach to the same problem. Children who present level 3 solutions will be praised and acknowledged, but their solutions may be too distracting for the children who are practising levels 1 and 2.

TABLE: Examples of expanded calculation

ADDITION	**25 + 18 =**	
Level 1 (count all)	= 20 + 5 + 10 + 8 = 20 + 10 + 5 + 8 = 30 + 13 = 30 + 10 + 3 = 40 + 3 = 43	(all numbers have been expanded – decomposition) (rearranged by tens and units) (added tens, added units) (decompose again if +13 is still difficult, or skip this step) (added tens) (added tens+units)
Level 2 (add on)	25 + 10 = 35 + 8 = 43 25 + 8 = 33 + 10 = 43	(leaving 25 unchanged, add tens) (then add units) OR … (start adding units, as the order is not important) (and then the tens)

Level 3 (regroup)	= 25 + 20 – 2 = 45 – 2 = 43	(find a clever way)

ADDITION	489 + 235 =	
Level 1 (count all)	= 400 + 80 + 9 + 200 + 30 + 5 = 9 + 5 + 80 + 30 + 400 + 200 = 14 + 110 + 600 = 14 + 710 = 724	(all numbers have been expanded) (rearranged by units, tens and hundreds, in any order) (added units, added tens, added hundreds) (added tens and hundreds) (done)
Level 2 (add on)	489 + 200 = 689 + 30 = 719 + 9 = 728 489 + 9 = 498 + 30 = 528 + 200 = 728	(leaving 489 unchanged, add hundreds) (then add tens) (then add units) OR … (start adding units, as the order is not important) (then the tens) (then the hundreds)
Level 3 (regroup)	= 489 + 11 + 224 = 500 + 224	(find a clever way)

SUBTRACTION	145 – 68 =	
Level 1 (count all)	LEVEL 1 IS NOT USEFUL FOR SUBTRACTIONS	(splitting 100 + 40 + 5 – 60 – 8 would work, although one would expect 100 + 40 + 5 – **60 + 8**, which needs brackets to be correct; furthermore 5 – 8 doesn't go. This is a big muddle, especially for primary children)
Level 2 (add on)	145 – 60 = 85 – 8 = 77 145 – 8 = 137 – 60 = 77	(leaving 145 unchanged, take away tens) (then take away units) OR … (start taking away units, as the order is not important) (then the tens)
Level 3 (regroup)	= 145 – 70 + 2 = 75 + 2 = 77	(find a clever way)

MULTIPLICATION	12 x 46 =	
Level 1 (count all)	= 10 x 40 + 2 x 40 + 10 x 6 + 2 x 6 = 400 + 80 + 60 + 12 = 400 + 140 + 12 = 540 + 12 = 552	(all parts of 12 and all parts of 46 separated) (done the four multiplications) (adding results) (combine some more) (done)
Level 2 (add on)	= 12 x 40 + 12 x 6 = 480 + 72 = 550 + 2 = 552	(leave 12 together, as we know the table) (multiplied both) (adding results) (done)
Level 3 (regroup)	= 6 x 92 = 6 x 100 – 6 x 8 = 600 – 48 = 552	(find a clever way: half of 12, double 46)

ABOUT DIVISION

Only in few cases will the normal decomposition assist with doing a division, namely if we divide by numbers that relate easily to 10. In the example 165 ÷ 5 it is helpful to regard 165 as 100 + 60 + 5, after which we see that 100 = 20 x 5, 60 = 12 x 5 and 5 = 1 x 5, therefore the answer is 20 + 12 + 1 = 33.

However, if we want to calculate 165 ÷ 3, the decomposition 100 + 60 + 5 doesn’t bring us much further.

Rather than decomposition in hundreds, tens, units, etc, divisions ask for decomposition in parts of the bigger number that can be divided by the number used in for division. The 165 can be split into 150 + 15 which is useful as 150 = 50 x 3 and 15 = 5 x 3 so the answer is 50 + 5 = 55.

Knowing this way of decomposing into dividable parts, offers children a way to tackle divisions with larger numbers. The dividable parts can be chosen freely, according to the child’s ability to find suitable numbers, e.g. 165 could also be 90 + 60 + 15 or 120 + 45. In the first case the result of the division becomes 30 + 20 + 5 = 55 and in the second case 40 + 15 = 55.

The above principle is used in the following approach to long division, which, as the new vertical approach to the operations (page 8), uses real numbers instead of digits.

TABLE: Long Division using Real Numbers

DIVISION	1761 ÷ 3 =			
	587	(we might as well decide against copying the answer on top)		
	3 | 1761		or just write:	Note that the choices (1200, 300, 210, etc.) were made freely by a child who felt confident about these multiples. Other choices could have been made and there is no need to work in order from the hundreds to units. After 561-300=261 the next step could be 7 x 3 = 21, so 261 – 21 = 240. 80 x 3 = 240 which then brings the question to its end.
	<u>1200</u>	= 400 x 3	400	
	561			
	<u>300</u>	= 100 x 3	100	
	261			
	<u>210</u>	= 70 x 3	70	
	51			
	<u>30</u>	= 10 x 3	10	
	21			
	<u>21</u>	= 7 x 3	7	
	0	+	+	
		total 587 x 3	587	

Please refer to the paragraph about remainders in divisions on page 18.

DO WE STILL TEACH THE VERTICAL ALGORITHMS?

The digit-by-digit, vertical algorithms that in the past formed most of what primary children learnt about doing calculations, were of course designed for a purpose. Their systematic approach enables us to calculate even with complex numbers, such as lots of decimals, where the expanded calculation would become clumsy and laborious.

It will therefore still be useful for primary children to learn to do these algorithms, but – due to the principle of focussing on UNDERSTANDING maths rather than learning tricks – the vertical approach should only be introduced after the children have thoroughly understood and practised the approach referred to as ‘expanded calculation’ (page 5).

Furthermore, the vertical approach does NOT have to go digit-by-digit! It is possible to calculate vertically while still using the real numbers instead of digits, as the next paragraph (page 8) will show. Certainly as an introduction, the vertical approach with real numbers below is strongly recommended!

The traditional, handy, digit-by-digit vertical algorithms will thus be the <u>final</u> version of the children’s approach to number problems, which has been built upon a number of years of experience using a full understanding of the number work. By now, children will be able to understand what the digit-by-digit trick is in fact doing in terms of the real numbers. We have thus avoided the problem of the past, that the children seemingly learned to make calculations, but did so by using the tricks, without really understanding what was happening.

The conversion from expanded calculation to real-number vertical operations fits very well in the class 4 curriculum, in the light of the children’s developmental stage, followed in class 5 or 6 by the digit-by-digit vertical algorithms. Still, once the digit-by-digit algorithms have been introduced, the teacher should regularly verbalise the operations as real numbers, to remind the children that each digit stands for something, a number of hundreds, tens, units, etc.

A VERTICAL APPROACH USING REAL NUMBERS

The following approach to vertical calculations is very closely linked to what is referred to as "expanded calculation". The numbers that were first decomposed and processed horizontally, will now be written vertically, thus preparing children towards the final step of the digit-by-digit system.

In the examples below, the same numbers and operations as above in page 5 are used, to show how the expanded calculation and the new vertical approach are closely linked. Again, the Level 1 (count-all) provides the best chance for children to fully understand the concept, while Level 2 (add-on) and Level 3 (regroup) make the calculations more efficient. Level 3 examples will not be included again, as they logically follow the numbers used above.

TABLE: Examples of the vertical approach using real numbers

ADDITION	**25 + 18 =**	
Level 1 (count all)	25 + 18 (20 + 10 =) 30 (add tens) (5 + 8 =) 13 (add units) (30 + 13 =) 43 (join the results) OR: (5 + 8 =) 13 (20 + 10 =) 30 (30 + 13 =) 43	The sum can initially be written out completely, including the (20 + 10 =), but soon, one would only write the added numbers down. As in expanded calculation, the order (tens and units) is not important.
Level 2 (add on)	25 + 18 (25 + 10 =) 35 (add the tens) (+ 8 =) 43 (add the units)	This time the first number 25 is not decomposed, but used unchanged. The order (tens and units) is not important.

ADDITION	**489 + 235 =**	
Level 1 (count all)	489 + 235 600 (add hundreds) 110 (add tens) 14 (add units) 724 (join results)	If hundreds, tens, units are still too complex to join easily, a further stage of rearranging can be added between the first step and the result, e.g. (find all the hundreds) 700 (find all the tens) 20 (find all the units) 4 724
Level 2 (add on)	489 + 235 689 (489 + 200) 719 (+30) 724 (+5)	

SUBTRACTION	**145 – 68 =**
Level 1 (count all)	LEVEL 1 IS NOT USEFUL FOR SUBTRACTIONS
Level 2 (add on)	145 – 68 85 (145 – 60) 77 (– 8)

MULTIPLICATION	12 x 46 =
Level 1 (count all)	46 x 12 (10 x 40 =) 400 (10 x 6 =) 60 (2 x 40 =) 80 (2 x 6 =) 12 400 140 (add the tens separately to manage the addition) 12 452
Level 2 (add on)	46 x 12 (12 x 40 =) 480 (use knowledge of 12 times table) (12 x 6 =) 72 552

The record of what was done is indicated above in brackets, e.g. (10x40=). This may be what the teacher initially expects, to be able to see what a pupil has done, but soon only the temporary results towards the answers should suffice.

The real number approach also works for complicated numbers

As one of the advantages of the traditional digit-by-digit approach was already mentioned that the system applies to simple as well as complicated calculations, including decimal numbers. However, without arguing against the usefulness of the old system when numbers become complicated, it must be said that the vertical approach with real numbers (see above) also works well for decimal numbers. In the table below, some examples are given.

By the time children are able to work with these complex numbers, in the higher primary classes, they will probably have been introduced to the digit-by-digit approach and regard this as a more economical way to calculate answers. However, it will be good for them to see that the trick that the digit-by-digit system is performing is in fact based on calculations with real numbers. Especially after several years of practising calculations with real numbers, they will be able to see the connection between these numbers and the handy digit-by-digit tricks.

Unfortunately, divisions that include decimals are the most cumbersome to consider from a real-number perspective. The mathematical principles do apply, but one has to go as far as saying for instance: “How many times can you fit the number 3 in 0,6 (six tenths)?” which is an awkward question to which the awkward answer is “The number 3 fits 0,2 times (two tenth times) in 0,6”. The mathematical explanation for this could be that if one considers how many times the number 3 fits in 6 (being ten times 0,6), our answer in this case must be a tenth of 2, as what we have to divide is a tenth of 6. This way of reasoning will, however, convince very few children and by the time children encounter divisions with decimals, they will manage better with the old-fashioned long division.

TABLE: Examples of the vertical approach including decimal numbers

ADDITION	25,37 + 18,965 =
Level 1 (count all)	25,37 + 48,956 (20 + 40 =) 60 (add tens) (5 + 8 =) 13 (add units) (0,3 + 0,9 =) 1,2 (add tenths) (0,07 + 0,05 =) 0,12 (add hundredths) (0,006 unchanged =) 0,006 (add thousandths) 74,326 (join them all)
Level 2 (add on)	25,37 + 48,956 (+ 40 =) 65,37 (add tens) (+ 8 =) 73,37 (add units)

	(+ 0,9 =) 74,27 (add tenths) (+ 0,05 =) 74,32 (add hundredths) (+ 0,006 =) 74,326 (add thousandths)

SUBTRACTION	**145,24 – 68,795 =**
Level 1 (count all)	LEVEL 1 IS NOT USEFUL FOR SUBTRACTIONS
Level 2 (add on)	145,24 – 68,795 85,24 (– 60) 77,24 (– 8) 76,54 (– 0,7) 76,45 (– 0,09) 76,445 (– 0,005)

MULTIPLICATION	**12,7 x 46,3 =**
Level 1 (count all)	46,3 x 12,7 (10 x 40 =) 400 (10 x 6 =) 60 (10 x 0,3 =) 3 (2 x 40 =) 80 (2 x 6 =) 12 (2 x 0,3 =) 0,6 (0,7 x 40 = 7 x 4 =) 28 (0,7 x 6 = 7 x 6 ÷10 =) 4,2 (0,7 x 0,3 = 7 x 3 ÷100 =) 0,21 400 (add hundreds) 170 (add tens) 17 (add units) 1,0 (add tenths) 0,01 (add hundredths) 588,01 (join all)
Level 2 (add on)	WHEN BOTH NUMBERS ARE TOO COMPLEX TO LEAVE UNCHANGED, LEVEL 2 IS NOT SUITABLE FOR MULTIPLICATIONS

THE THREE LEVELS OF OPERATIONS

Learning a new mathematical skill, whether it is in early primary, higher primary or even in High School, most pupils will first need to break down a mathematical operation into many small parts, whereas with some experience with the new operation, pupils grow to higher levels of competence. For the teacher it is important to know which step level they can manage, and the levels also guide the teacher how to introduce a new topic and what to expect after a while of practising.

Level 1 – Count all

The Count-all level refers to the activity of counting objects in class 1, but also to the introductory stages, in higher grades, for instance where numbers are fully decomposed into hundreds, tens and units in order to make the operation accessible. This level helps children build a full understanding of the mathematical principles used in the operation they are practising.

Level 2 – Add on

Quite soon, however, the pupils will find out that there is a quicker way to do the operation than splitting everything up and then having to count it all together again. This leads to a new level of doing the operation more effectively. Usually at this level, one number is kept unchanged, while the other(s) are decomposed to make the operation manageable.

Level 3 – Regroup

Finally there are sometimes even cleverer solutions for the same problem, which require a quite advanced level of thinking and understanding. This level is first reached by the mathematically gifted children in the class, who find a way of rearranging parts of the operation in a way that leads to an easier solution, but is based on clever manipulation and mathematical insight.

The table below contains examples to show how the three Levels of Operations apply to almost each new mathematical concept pupils encounter, and how pupils move up from one level to the next, once they familiarise themselves with the concept.

It is important to realise that almost all children start from level 1 each time they learn something new, and that it depends on the children’s mathematical abilities, as well as on their temperaments, how quickly they get bored of doing things the slow way, and how soon they find out that it can be done more cleverly.

What to aim for?

Although most pupils naturally reach their first workable level of skills at level 1 (count-all), this is often a rather clumsy and laborious way of solving the maths problem. In most cases, the aim should be to reach at least level 2 (add-on), especially before moving on to something more difficult. Level 3 shows that the pupil has really mastered the ability to find clever solutions for the problem, and one would hope that all pupils reach this stage even if this happens a year later than the year in which they reached level 2. It is therefore useful to go back to the maths work of a previous grade from time to time, and see “how much cleverer” the pupils now are in solving these problems.

How to use these levels in our teaching?

The progression from the first to the higher levels is in fact a natural process, in which pupils could determine their own moments for taking learning steps. The large majority of pupils will not remain happy with doing things at level 1, and sooner or later they will discover level 2 or even level 3. This own discovery is much more effective for pupils than steps they have been taught.

The practicalities of dealing with a class of children, always at various levels of ability, complicate the teacher’s choices in what to teach and at what level. A teacher will invariably need to demonstrate ways of solving mathematical questions.

Being aware of the three levels will already help the teacher prepare a logical, step-by-step approach towards the level that is envisaged as the outcome of a series of lessons. If this is at level 2, which it should normally be, level 1 is where it should start. Accurately following all pupils’ level of competence is highly important – assuming that “the class” has grasped something, on the basis of some clever responses from clever pupils, is a crucial mistake. Investigate thoroughly how each pupil manages, and at which level.

The principle that mathematical problems can be solved in various ways can be used as soon as pupils experience operations. During the maths lessons, it should be clear that any correct solution is acceptable, not only the method explained by the teacher. In fact, the different ways of solving a question could often enliven the lesson, keep children alive in their thinking, and stimulate the easy learners to experiment.

TABLE: Examples of the Three Levels of Operations

	Level 1 **Count all**	**Level 2** **Add on**	**Level 3** **Regroup**
8 + 7	Count out 8 counters, then another 7, then add them together and count all 15	Count out 8 counters, then another 7, then count on from 8 until 15	Realise that 8 + 2 make 10, then count on the remaining 5 to get 15
28 + 14	Add 20 + 10 + 8 + 4 = 30 + 12 = 42	Do not break up 28, but add on 28 + 10 = 38, then + 4 = 42	Realise that 28 + 14 = 30 + 12 (two from 14 move to the 28) = 42

19 x 12	Continue table of 12 until 19 x 12, or split everything up: 10 x 10 + 10 x 2 + 9 x 10 + 9 x 2 = 100 + 20 + 90 + 18 = 228	Leave one number unchanged, e.g. 12 10 x 12 = 120 9 x 12 = 108 120 + 108 = 228	Realise that the answer is 1 x 12 less than 20 x 12, so 240 – 12 = 228
65 ÷ 5	Write a list of multiples of 5 until reaching 65	From knowing 50 ÷ 5 = 10, continue to 55 ÷ 5 = 11, 60 ÷ 5 = 12, 65 ÷ 5 = 13, or: 65 consist of 50 + 15, of which 50 ÷ 5 = 10 and 15 ÷ 5 = 3, so 65 ÷ 5 = 10 + 3	Realise that 65 ÷ 5 is the same as 130 ÷ 10 = 13
20% of R150	1% of R150 is R1,50 20% of R150 = 20 x R1,50 = 20 x 1 + 20 x 0,50 = 20 + 10 = R30	10% of R150 = $\frac{1}{10}$ x 150 = 150 ÷ 10 = 15 so 20% of R150 = 2 x 15 = R30	20% x R150 = 10% x R300 = R30

THE THREE LEVELS OF VISUALS

Level 1 – Hands-on, concrete materials

Handling real items is the most effective way of learning mathematics. In class 1 children should be working with counters to practise counting and operations. Later, children should cut circles when they start learning about fractions. Still later, they should measure a real piece of ground to understand perimeter and area. The more practically active, and the more their various senses are involved, the better children learn the principles of mathematics.

Level 2 – Diagrams and pictures

Once children have experienced mathematical concepts practically, the next step is to remind the children of this experience with a diagram. The visual reminder will help the children connect what they are attempting to their previous experience, thus furthering their level of understanding to a more inner level.

Level 3 – No visuals

Eventually the mathematical concepts can be understood without the help of visual representation. Children can recreate the aspects that were visualised earlier as a thought process, and they are able to come to conclusions about the mathematical operations in an abstract way.

What to aim for?

In most cases, the aim is to build mathematical understanding beyond the level of visual aids. It is important to realise, however, that too easily “mathematical understanding” is misinterpreted as “having learnt a set of rules that the child can implement”. A child who is able to answer that the area of 3m by 4m is 12 m^2, but has no clue about what such area would mean in a vegetable garden, has in fact very little mathematical understanding. Such understanding only exists if the child is fully able to relate the abstract to the visual levels 1 and 2.

How to use these levels in our teaching?

Introducing new mathematical concepts in a practical, hands-on way (level 1) is clearly the most effective way of building understanding in children. After that, the children will relate well to drawings and diagrams representing the same concepts. These drawings will still assist the children for quite a while, on their way to abstract understanding. And whenever a child struggles to master level 3, a quick revision of the diagram stage will be exactly the reminder this child needed.

THE THREE LEVELS OF PRACTISING

Level 1 – Class as a whole

Many forms of practising mathematical (and other) skills in class are done class-wise. Learning to count numbers, reciting multiples and multiplication tables, bonds and many more learning moments are shared by the group as a whole. This way of practising is effective in terms of time, and supportive of the weaker children as they are pulled through by the children who learn faster.

In addition to wondering how long the class-wise practising remains interesting for the quicker children, the main question about this level of learning is how to find out who has indeed learnt something and who hasn't. It is difficult to distinguish these differences clearly in a big group, and especially the quality and accuracy of what each child is saying remains a question.

Some teachers' teaching approach also uses the whole class as a sounding board, for instance when they ask the class a question and move on to the next topic when some child has answered. This is a dangerous approach as receiving an answer from a quick learner does not at all mean that the class as a whole is ready to move on.

Level 2 – Parts of class

Alternating groups within the class (half of the children or smaller groups at a time) is a useful next step, after the whole class has practised something. The children in the active group will feel more challenged than they would have been in the whole class, as the group is smaller and less supportive, and as there are now listeners checking how well the group is performing. The children who are quiet for a little while have a moment to learn in a different mode: from listening to others doing the exercise, while checking consciously or unconsciously whether what the others are doing matches with the child's own skills and understanding.

The teacher will already be able to observe much more detail in what individual children are managing or not. The smaller the groups become, the more detail the teacher will observe. Some exercises can be performed by two children at a time.

Level 3 – Individuals

Once the class and smaller groups are able to show good results, many exercises can be performed by one child at a time. This is the moment when individual children can show that they master the counting, the multiples, or the series of maths operations on their own. The teacher will clearly observe the individual skills in the children.

It is of course important not to exert pressure on individual performance, and to avoid letting children experience failure and disgrace, especially in the lower grades, but in a playful, natural way children should develop the ability to show individually what they have learnt.

Including this individual level of competence is an important factor in giving meaning to the class-wise and group-wise practising at level 1 and 2. Reaching the individual level of competence enhances the children's experience that the learning process had a purpose, which will in turn improve their motivation to take on practising the next task.

THE THREE LEVELS OF COMPETENCE

Level 1 – Ability to find a solution by any means

A class 1 child who counts out two different numbers of counters, then puts them all together and counts the total, or a class 4 child who writes out the table of 13 to answer 5 x 13, or a class 6 child who finds 25% of R 160 by calculating 25 x R 1,60, all these children find solutions at level 1, which means that some form of writing or practical arranging is used in order to find the answer to a question.

Level 2 – Ability to find a solution mentally

A mental calculation to find an answer means taking the practical or written level away, imagining the numbers in one's mind and performing a calculation with them. In class 1 a child might think about 6 + 4 for a while, and after some inward counting, respond with '10'. A class 3 child practises sums like 27 + 14 mentally, for instance by first adding 10 to the 27 and then 4. A class 7 child may consider a question with negative numbers, and come with an answer after a mental calculation.

Level 3 – Ability to respond instantly

This level is acquired after much practising, enabling the child to respond without having to think. Reaching this level with simple mathematical elements like bonds and multiplication tables is essential for a child to be able to tackle mental calculations in which several mathematical steps should lead to an end results. The instant responses to the basic elements will enable the child's mind to keep track of the bigger picture of the problem. Children who do not have the instant responses will lose the line of the total problem by having to actually calculate these basic elements.

The multiplication tables form an obvious bulk of work that needs to be covered to reach this level. In addition to knowing the multiplication tables as "this times that", the corresponding divisions should also be known equally well. Therefore, once children master 1 x 4 = 4, 2 x 4 = 8, 3 x 4 = 12, they should also practise instant responses to questions such as 4 ÷ 4 = 1, 8 ÷ 4 = 2, 12 ÷ 4 = 3.

Besides multiplications tables, the bonds up to 20 are an equally important set of questions that children should practise towards the level of instant response. Any combination of two numbers adding up to a maximum of 20 should be known instantly. Bonds are practised as additions (e.g. 12 + 4 = ???) as well as subtractions (e.g. 19 – 6 = ???).

In addition to multiplication tables and simple bonds, the level of instant responses should be reached when calculating with multiples of 10 or 100, which is often necessary in finding step-by-step mental solutions for more complex calculations. Practise things like:

26 + 10, 26 + 20, 26 + 30, 26 + 40, etc.

10 + 45, 20 + 45, 30 + 45, 40 + 45, etc.

10 x 4, 20 x 4, 30 x 4, 40 x 4, etc.

1 x 30, 2 x 30, 3 x 30, 4 x 30, etc.

5 x 10, 5 x 20, 5 x 30, 5 x 40, etc. (at a later stage, as the number of groups is now changing – see what was written about the order of multiplications)

Doubling and halving are also important skills to have without needing time to think. Similar to their equivalents of $\boxed{2 \text{ x} \ldots}$ and $\boxed{\ldots \div 2}$, they can be practised as in the following examples:

double 10, double 11, double 12, double 13, etc.

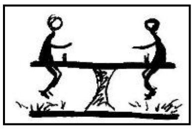
half of 10, half of 12, half of 14, half of 16, etc.

Lastly, a special 'instant response' can be developed around the number 9. Adding 9 to any number increases the number by 10 but minus one, which shows in the reduced last digit, e.g. 43 + 9 = 52, 75 + 9 = 84. The opposite happens when the number 9 is subtracted, e.g. 63 – 9 = 54, 48 – 9 = 39. Practising series of +9 and -9 will help the children in many future moments of calculating with that tricky number 9 (tricky, because it is almost 10 but not quite).

THE EQUAL SIGN

It is an important but far from easy principle for young children to learn that the equal sign represents a balance. What is happening on one side of the equal sing has the same numerical value as on the other side. This understanding can be enhanced by including in one's maths teaching the image of a balance.

In the way teachers introduce the symbols of mathematics to young children, a suitable image for the equal sign is as important as the images often used for the symbols representing the four operations. Helen O'Hagan's image of the see-saw is an excellent choice: What happens on the left hand side balances with what happens on the right hand side. This principle can be put into practice on a real see-saw, besides the problem that some children are heavier than others and that a child's position on the see-saw (e.g. closer to the centre) also determines which way the see-saw goes.



After the introduction of the equal sign as an indicator of balance, much practice will be needed for children to keep regarding the equal sign in that way. Children who insufficiently practise the balance principle tend to regard the equal sign as the indicator that they need to come up with an answer to the operation.

The main factor contributing to a good understanding of the balance principle is to vary the format of sums, i.e. to avoid a standard format like 5 + 3 = ??? but to freely move between various possibilities of the position of the operation and of the open space where the answer is to be filled in. This should include the 'normal' situations with one operation and one single answer (result of the operation), as well as different operations left and right. Examples of possibilities follow in diagram below.

As a teacher’s way of addressing the different formats of sums can make or break the children’s understanding, the possible wording of each example has been included. Although there are various options for indicating the open space (for the expected answer), for instance ….. or ___ the box $\square$ is used here as this is a recommended way of visualising the space for the answer. Dots and dashes can easily be misinterpreted as maths symbols, especially when the work is not very neat. Ensure that a box is big enough to fit the answer easily.

5 + 3 = $\square$	$\square$ = 5 + 3	which number is equal to 5 plus 3 ?
5 + $\square$ = 8	8 = 5 + $\square$	which number do we need to add to 5 to make 8 ?
$\square$ + 3 = 8	8 = $\square$ + 3	which number makes 8 if we add 3 to it?

12 + 3 = 10 + $\square$	what do we need to add to 10 to get the same as 12 + 3 ?
12 + $\square$ = 10 + 5	what do we need to add to 12 to get the same as 10 + 5 ?
$\square$ x 4 = 7 + 5	how many times 4 is equal to 7 plus 5 ?
3 x 10 = $\square$ x 6	how many times 6 is equal to 3 times 10 ?
24 – $\square$ = 2 x 9	what must we take away from 24 to get the same as 2 times 9 ?

However, it is important to keep in mind that the equal sign’s balance principle is only about the numerical balance. The equal sign disregards the fact that the four operations are very different in their qualities. Doing the operation 4 + 3 + 2 + 1 gives the total 10, and so does making 2 groups of 5. Yet, the qualities of how we made 10 in each case are very different. 4 + 3 + 2 + 1 = 2 x 5 is only true for its quantities, not for the processes that happen on both sides of the equal sign. Thus, children will have to learn that the equal sign is a numerical balance, regardless the processes that happened to come to this balance.

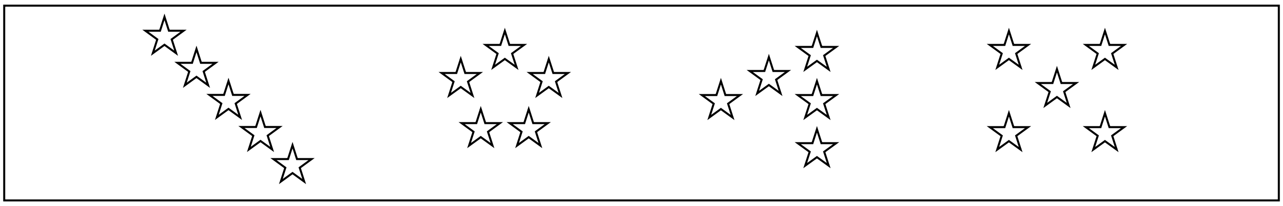
More specifically, the equal sign has the problem that it is unable to show what has happened over time. In story sums, leading to an end result, there is often a before and after experience. This experience, however, is caused by the operation that took place, and the equal sign is in fact a clumsy element of the story. For instance in the story of the woman who went to market with 7 oranges, sold 4 of them and returned home with 3, the real activity is found in the subtraction, which causes the before and after difference. The equal sign here takes an awkward position, as on the left there are big numbers (7 and 4) which seem to balance with the 3 on the right. This problem occurs with subtractions and divisions, whereas additions and multiplications are easier to understand with the balance image.

The equal sign is often used incorrectly – especially when working with expanded calculation – in the sense that after the equal sign a temporary answer is written, after which the child (or the teacher) continues with the problem that had been broken down into various steps. The following examples will hopefully clarify which ways of using the equal sign are incorrect, and how these situations can be avoided. At all times, there needs to be a balance between what appears on the left and the right side of the equal sign.

Using the Equal Sign correctly		
38 + 26 =	38 + 20 = 58 + 6 = 64	The sum has been solved correctly, but the equal signs are incorrect. There is no balance between 38 + 20 and 58 + 6.
	38 + 20 = 58 58 + 6 = 64	Instead, writing needs to stop when the balance is complete, and to be continued on the next line.
	38 + 20 = 58 + 6 = 64	Alternatively, the repetition of the 58 can be omitted and the next line starts with +
174 – 95 =	174 – 100 = 74 + 5 = 79	Incorrect, there is no balance between 174 – 100 and 74 + 5
	174 – 100 = 74 + 5 = 79	Correct. The second line still means 74 + 5 = 79 and there is balance.

SUBITIZING – Recognising quantities at a glance

Recognising quantities (up to 6) without actually counting them (such as the numbers of dots on dice) is an important skill for children to develop. Playing dice games will help them with these patterns is a good approach. However, patterns of dots in other shapes than the ones on dice are also worth practising. Some examples other than the patterns on the dice:

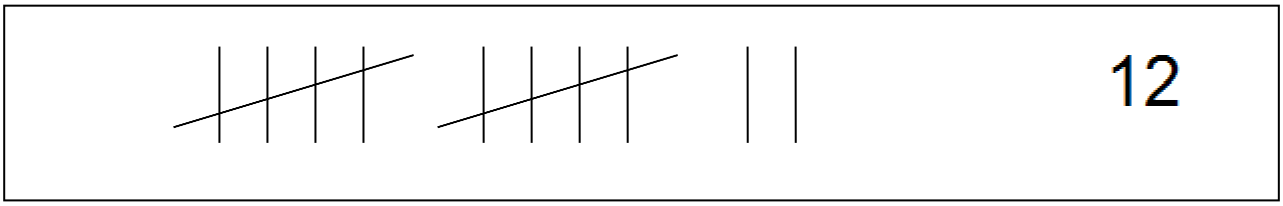


TALLYING – Counting all in groups of five

For children learning about our number world, tallying is a very useful activity. It practises the principle that each unit of something can be represented accurately by a written record, while the written records are well organised in groups of five, which makes the final results much easier to interpret as well. Many activities can be created where a record of achievement or a record of observations is kept by tallying.

The groups of five already start to make the children aware of the multiples of five and their connection to the multiples of ten.

Matching the tallies with the numbers the children are learning, as well as with the numbers on dice, will greatly enhance their ability to handle numbers and quantities with confidence.



MULTIPLES AND TABLES

Multiples are best drawn from counting in a rhythm. This rhythm should be supported with some kind of movement, in which there is a natural moment emphasizing the multiples that are practised. A movement is helpful in support of speaking the multiples if it is not too complicated (ie. distracting from the number work), and not so easy that the children lose interest in making the movement.

After saying all the numbers, the numbers that aren't multiples are first whispered and then left out altogether, while the rhythm still supports where those numbers used to be. The sequence of 3 then becomes * * 3, * * 6, * * 9, etc. A short pause after each multiple is quite acceptable.

Once the multiples have been practised within the rhythm, the exercise moves from counting all to saying the multiples only. The sequence of 3 then becomes 3, 6, 9, 12, etc. with another movement which may be linked to the previous, counting movement.

The new movement, for the multiples only, could well emphasize the numerical pattern that can be found in most tables. The multiples of 6, for instance, end on 6, 2, 8, 4, 0 as in 6, 12, 18, 24, 30, 36, 42, 48, 54, 60. This particular series would be emphasized through a series of 5 movements (e.g. head / shoulders / hips / knees / toes) after which the 5 movements repeat for the next 6, 2, 8, 4, 0.

Once the children know the multiples, the table can be practised. The difference between the multiples (3, 6, 9, 12, etc) and the multiplication table (3 = 1 x 3, 6 = 2 x 3, etc) is important for the class to be conscious of. It is especially vital for children's ability to connect properly to these multiplications that they are done in the correct order of $\boxed{\text{increasing number}}$ x $\boxed{\text{multiplication table}}$ = $\boxed{\text{result}}$.

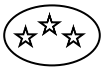
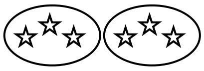
When starting with the result (because of the principle of starting with the whole, the order is $\boxed{\text{result}}$ = $\boxed{\text{increasing number}}$ x $\boxed{\text{multiplication table}}$, and it is a good habit to include both ways in the early stages of practising a table (e.g. 3 = 1 x 3, 1 x 3 = 3, 6 = 2 x 3, 2 x 3 = 6). The slower children have the chance to hear each new step in the ladder, and to actively speak the returning version of it.

As an example, the correct format for the table of 3 is:

3 = 1 x 3	1 x 3 = 3
6 = 2 x 3	2 x 3 = 6
9 = 3 x 3	3 x 3 = 9
12 = 4 x 3	4 x 3 = 12

The correct order of a multiplication supports the understanding that multiplications are about counting groups of things. 4 x 3 means four **groups of** three; the multiplication sign and the phrase ‘groups of’ should be interchangeable. The fact that the numerical value of 4 x 3 is the same as that of 3 x 4 is something the pupils will easily discover at some stage, but at the stage of learning a table, the correct order is very important and swaps will only cause confusion.

The learning of the multiplication tables will benefit from drawing the groups.

E.g. 1 x 3 = , 2 x 3 = , 3 x 3 = , etc.

After practising the reciting plus movements, the multiplication table must also be practised in writing. As revision, the multiples can also be identified on a number grid (e.g. a hundred square), which reveals the pattern of each multiplication table.

The aim for all this work is the children’s individual mastering of the multiplication tables, both as a full set that can be recited and written, but also at the level of answering any question in any order, e.g. 6 x 3 = ???, 9 x 3 = ???, 4 x 3= ???. This individual competence can be practised during movement time, e.g. after the completion of the group-wise practising, individual children take turns answering random questions from the table. The competence is also practised as written work.

Mastering multiplication tables is one of the skills children should require at the level 3 (competence: ability to respond instantly) see page 14.

As each multiplication has a corresponding division, and divisions should be mastered with the same level of competence, it is useful to turn a multiplication table that children have learnt into a division table, for instance:

1 = 3 ÷ 3	3 ÷ 3 = 1
2 = 6 ÷ 3	6 ÷ 3 = 2
3 = 9 ÷ 3	9 ÷ 3 = 3

Once children have learnt about fractions, they will benefit from practising some multiples and tables of fractions, such as:

$\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1,	$1 \times \frac{1}{3} = \frac{1}{3}$
$1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, 2,	$2 \times \frac{1}{3} = \frac{2}{3}$
$2\frac{1}{4}$, $2\frac{1}{2}$, etc.	$3 \times \frac{1}{3} = 1$
	$4 \times \frac{1}{3} = 1\frac{1}{3}$

HALVES AND DOUBLES

Half of an apple is a well understood concept long before children learn about fractions. Halving numbers, such as the number of children in class, or the books on the shelf, should be practised right from class 1. Soon after learning numbers (e.g. 20, 50, 100, 1000) they should have an idea of the half-way mark.

Number lines are very helpful in visualising what it means to take half of something. After writing the numbers 1 up to 20 on a number line, it is easy to see that the numbers 1 to 10 are the first half, and the numbers 11 to 20 the second half. The division is between the numbers 10 and 11.

Odd numbers obviously create a difficulty (see also: remainders in division). Again the number line is helpful. To find half of 15, write the numbers 1 up to 15. The numbers 1 to 7 correspond with 9 to 15, and this time the half-way division is not between two numbers, but right on top of the number 8 in the middle. If the 15 numbers represent apple pies, we can now complete the half division by cutting number 8 into two halves and even young children can see that an equal share is now seven-and-a-half. This is not a situation to avoid, even in class 1.

An odd number of children or books of course leads to the predicament that the child in the middle cannot be cut into two halves (neither could the remaining book). Here the children will be learning that in such cases an even number will be required to make the halving possible without someone having to step out.

All these deliberations about halving are useful mathematical experiences, even for the young child.

Doubling is in fact the same as multiplying by two. However, the principle of doubling is an important concept for children to practise with extra emphasis. It will greatly enhance their mathematical understanding to practise doubling thoroughly, within the number range they work with in their grade.

REMAINDERS IN DIVISION

As in the paragraphs about halving, above, there are divisions that produce a straight-forward answer, and ones that end with a remainder.

Some teachers of mathematics argue that children should get used to both types of divisions right from the beginning. However, ending a division with a remainder and having to consider what that means is quite a confusing moment for children. Combining the remainder questions with the process of grappling with the concept of division as such, is therefore not my choice.

During the process of learning to do division, which is usually the most difficult operation to let children find a connection to, the teacher makes sure that all division questions will produce a straight-forward answer. This is the case when the division is the opposite of a multiplication the children have learnt, e.g. $4 \times 6 = 24$ turns into $24 \div 6 = 4$.

Only when the children have fully understood the process of division, and are able to work out the answers (derived from the opposite of multiplying), will they be introduced to the problem of the remainder. This will most likely be during class 2 or 3.

Various versions of “Remainder Games” exist, in which a whole class freely walks around until the teacher calls out a number, by which the children need to group themselves. Choosing different numbers leads to different remainders, being the children who were left out. After having played such games, teacher and children can reflect on this problem in a maths lesson, so that the concept of the remainder in divisions is understood.

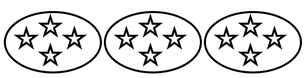
However, although a remainder in some situations is acceptable (children or books cannot be cut in halves or smaller parts), quite often it is possible to cut up the pancakes or apple pies that are in the remainder, and let children share these parts. This important aspect of the remainder is often completely forgotten. Even before children have been formally introduced to fractions, they are aware of halves and quarters. A class 3 child can already understand that 6 apples divided by 4 children gives them each one whole apple, after which the remaining 2 can be halved and shared, which give each child a total of an apple and a half. Once children know about fractions, they should soon learn that if $20 \div 4 = 5$, the question $21 \div 4$ also has an exact answer. The 4 people who share 20 each receive 5 whole items, after which the remaining item is also shared, so each of the 4 people receives another quarter, giving the final result of $5\frac{1}{4}$ for each person.

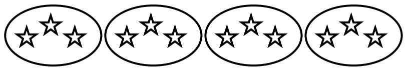
The concept of dividing the remainder further, into fractions, can and must be fully utilised once the children are ready to understand these fractions. Using the remainder is only a temporary stage in mathematics teaching, which should be replaced as soon as possible by the realisation that the division can be continued, thus producing a fraction.

COMMUTATIVE ASPECTS OF OPERATIONS

In two of the four processes (addition and multiplication), swapping the number that are being added or multiplied does not affect the outcome of the operation. At some stage, children will need to become aware of this, but we need to treat this commutative aspect of the number work very carefully.

As it was argued in connection with the order of multiplications, 3 x 4 is not the same as 4 x 3, although both result in 12.

3 x 4 means 3 groups of 4, which looks like 

4 x 3 means 4 groups of 3, which looks like 

Although this qualitative difference appears most strongly in multiplications, even in additions the order of the numbers that are being added is not irrelevant. Especially for children who are still in the process of connecting to the concept of addition (classes 1 and 2) the original number has a quality that is different from the number that is being added. It would be counterproductive in terms of the imaginative approach to the mathematical operations to carelessly swap the two numbers.

Some teachers are eager to spend teaching time on the principle of commutation. This is, however, one of those typical aspects of mathematics that children can discover for themselves. When a child is ready and makes the discovery, it will make a much bigger impact on the child than being taught about this principle before being keen and ready.

Once children start finding out that numbers can be added or multiplied in any order, the teacher can confirm this, and a good question in return, for the children to investigate, would be whether the ability to swap numbers occurs in all the operations. By trying out some examples, the children will easily find out that it does not work in subtractions and divisions.

In divisions, however, there is a commutative aspect that will quite soon be needed for children to solve their questions. The commutation here is between the divisor (the number in the middle) and the answer (the number after the equal sign) for instance in 21 ÷ 3 = 7 and 21 ÷ 7 = 3 . When we teach these divisions to children, using the concept of dividing equally, the first version would mean “divide 21 equally among 3 children and how many do they each receive?”, whereas the next version would mean “divide 21 equally among 7 children and how many do they each receive?”. Numerically, however, these divisions are equivalent. This aspect of commutation will be useful when later, in long divisions, children need to find out how many times a number fits. To answer the division 21 ÷ 3, the question becomes “how many threes are there in 21?” which in fact turns the “real” meaning of the division (three groups, how many in each group?) to “three in each group, how many groups?”. Fortunately, the question “how many threes are there in 21?” is quite an easy and natural response to a division problem, and many teachers don’t even notice that in fact commutation has started taking place.

PLACE VALUE AND DECOMPOSITION

The image that was often used in the past, of bundles of 10 and packets of 100, mainly aimed at changing one’s focus to the single digit that represents 10s, 100s, 1000s, etc. This single-digit focus was important for doing the vertical algorithms the old way, like this:

268 + 374 642	8 + 4 is 12. Write 2, remember 1. 6 + 7 is 13 plus the remembered 1 is 14. Write 4, remember 1. 2 + 3 is 5 plus the remembered 1 is 6. Write 6, nothing to remember.

As described in the paragraph about expanded calculation (page 5) this way of teaching the calcuations is no longer preferred, especially at the stage where primary children need to develop real understanding of the numbers and the operations involved. The single-digit focus in fact takes children away from what is really happening.

Instead of a focus on single digits, teaching and learning of maths should focus on the real numbers that are represented by the written form of the number, where of course place value is an important aspect. In the number 268 for instance, the number 2 represents 200, the number 6 represents 60, the number 8 is just 8, or one could say 8 normal numbers. The term ‘units’ doesn’t help children much with their understanding.

As a result, the approach to place value will need to emphasize the real numbers inside what was written. In our speaking, we emphasize this by verbalising the whole number correctly (see verbalising numbers on page 20), in addition to which we often verbally describe the elements contained by a number, e.g. 3642 consists of 3000, 600, 40 and 2. This principle of decomposition should also be practised regularly in writing, e.g. 3642 = 3000 + 600 + 40 + 2 .

How decomposition is used for doing calculations, will be explained in the chapters Expanded calculation (page 5) and A Vertical Approach with Whole Numbers (page 6).

Place value becomes extra important when children are learning about decimal numbers. The number 125.483 for instance consists of 125 (whole numbers) and 4 tenths and 8 hundredths and 3 thousandths. In the paragraphs about introducing decimals (page 25) the way of working with place value and decimals will be explained.

VERBALISING NUMBERS

It is important that children learn to verbalise numbers that they are doing calculations with, to avoid doing calculations with long numbers (e.g. long division) without having any clue what the long numbers represent. Obviously, children in the lower grades will verbalise numbers such as 12, 46 and 89.

Once there are hundreds and thousands in the number, it is still important to verbalise the number completely, instead of shortening it. For example: 158 sounds like one hundred and fifty-eight (not one fifty eight), 493 sounds like four hundred and ninety-three (not four ninety-three), 2012 sounds like two thousand and twelve (not twenty-twelve).

In the higher grades, the numbers with millions still need to be practised, to keep alive the pupils' understanding of the big numbers they are working with. For example: 34,895,172 sounds like thirty-four million eight hundred and ninety-five thousand one hundred and seventy two.

As described from page 25 about decimals, the initial way of verbalising decimal numbers should be using the tenths, hundredths, etc. The number 125.483 for instance sounds like 125 (whole numbers) and 4 tenths and 8 hundredths and 3 thousandths.

NUMBER LINES

Creating a number line can be done as a picture, but also in real size, when children step forward and backwards in space. The numbers can be freely created while they speak the numbers, but also indicated on the floor with chalk, or represented by children standing still with or without a number card in their hands.

Both as a drawing and as steps in space, number lines are helpful tools to practise the awareness of neighbouring numbers and steps up (addition) or down (subtraction). Stepping groups of distances, e.g. stepping two's or three's is a good representation of the concept of multiplication, while the opposite, dividing a bigger number into equal parts visually and/or spatially represents the concept of division.

NUMBER GRIDS

Working in number grids (e.g. the hundred field), for instance to discover the patterns of multiples, is a useful activity. It is an enjoyable moment for children to colour in the patterns they discover, relating to the various tables, and they may discover aspects of the tables that they haven't been aware of thus far.

However, working with number patterns on number grids is mainly useful as a form of consolidation. Learning through movement is a much more effective way for the children to realise the number patterns rhythmically, and to be able to practise them again and again without noticing the amount of repetition that is happening this way. What children learn by moving, remains a long-lasting skill, whereas an intellectual learning moment has a limited life-span.

THE NUMBERS 0 AND 1

Teachers must be very aware that the numbers ZERO and ONE usually cause an extra level of difficulty for children, when they try to understand what is happening.

For instance, understanding 2 x 3 as two groups of three is easier for young children than 1 x 3 (one group) or even 0 x 3 (no groups at all). When there are several groups, the concept of the group (ie. the repeating number in each group) is easier to see than with a single group, and no groups at all is yet a higher level of thinking. Many children answer the question 0 x 3 as 3.

Adding 0 to a number, or adding a number to 0 is also much more difficult for children than adding one number to another. In decimal fractions, a 0 behind the decimal point always causes extra difficulties.

Dividing by 1, and ending up with the same result, is much more difficult than dividing by a number that forms several smaller groups.

The numbers 0 and 1 therefore need an extra level of consciousness, both in the teacher and in the children. With sufficient guidance, they can surely get used to these special numbers. The number 0 can be verbalised as "nothing" and 0 x can be read as "no groups of …". This will enhance the children's understanding, as 14 + 0 will sound as "fourteen plus nothing", 5 x 0 as "five times nothing" and 0 x 6 as "no groups of six".

When practising the concept of division, which obviously starts with making more than one group, the time will need to come that children consider what happens if the division is by 1, which means that one receiver takes the whole lot.

THE ADVANTAGE OF NON-REPEATING NUMBERS

When children are being introduced to a new concept in maths, it is important to avoid repetition of numbers that might lead to confusion. When introducing the concept of addition, for instance, 3 + 4 is a better choice than 3 + 3, or 4 + 4. Repeating numbers can easily confuse children in understanding which is the starting number, and which the number that performs the operation. Similarly, in multiplication, 3 groups of 4 items gives children a clearer picture than 3 groups of 3, or 4 groups of 4. Dividing 25 by 5 gives the answer 5 – a reason for confusion, so dividing 21 by 7 is much clearer. In the first exercises towards understanding place value, children should see different numbers of hundreds, tens and units to distinguish easily between the different values.

Of course, children will soon need to be able to handle sums with repeating numbers, but at the stage of practising new concepts repeating numbers are to be avoided. Using different numbers helps the children to see exactly what happens with each element of the operation.

Teachers who make up sums while they are teaching regularly come to moments when they realise that the choice of their numbers has, in hindsight, not been very fortunate. It is therefore important to plan all sums as part of one's lesson preparation, including a consideration of how the answer is calculated.

DIFFERENT STRATEGIES – CHILDREN AS INVESTIGATORS

There are often several correct ways of solving a mathematical question. In the past, the teacher would choose one approach and expect all children to use that approach for finding their answers. Merely following the teacher's instructions is, however, the least effective way for children to develop a lively understanding of mathematical concepts.

Obviously, a teacher will need to choose an approach, suitable as an introduction of a new concept, when working with a class towards a new skill in maths. This approach will probably be at level 1 of handling the operation (see page 4). However, the children should be encouraged to find other ways towards the correct answer, and a classroom atmosphere of welcoming different approaches is very important. The old expectation that all children should be using the teacher's instructions literally is not conducive to the development of the children's thinking skills. Instead, the children should be aware that there are often various valid ways to come to an answer. Asking children "how did you get to your answer?" and comparing responses, encourages children to use their thinking skills instead of merely putting a learnt routine into practice.

FROM THE WHOLE TO THE PARTS

Rudolf Steiner's advice to work from the whole to the parts has, especially in mathematics, led to several different interpretations. There are Waldorf teachers who, because of this advice, always stick to the order XXX = XXX + XXX in all the sums they write. One should, however, wonder what it really is that Steiner has indicated here.

In the Ilkley course (chapter IX) Steiner gives the following three examples:

> *"Here are 14 balls. Now we divide them – here we have 5, here 4, here 5 again. Thus we have separated the sum into 5 and 4 and 5."* - this forms the sum 14 = 5 + 4 + 5 .
>
> *"If I have 7, how much must I take away to get 3 (instead of: What remains over if I take 4 from 7*)?" – this forms the question 7 – $\boxed{}$ = 3 .
>
> *"We should not ask what will result when we divide 10 into two parts, but: How must we divide 10 to get the number 5."* – This forms the question 10 ÷ $\boxed{}$ = 5 .

From these examples we can conclude that starting with the whole does not mean that the end result, the outcome of the operation, is to be regarded as the whole. Steiner starts his examples with the biggest number, the total of the items involved. From there he moves into the parts. He finds possible additions, and possibilities to subtract or subdivide.

What emerges from these examples is mainly, that Steiner considers the whole of the process, from a starting number to an end result, with a special focus on the operation that changed the starting number to the end result. This is clearly a strong plea to avoid (at the time of introducing mathematical concepts) the order XXX + XXX = $\boxed{}$, but equally strongly opposes the thought that the end result should always be written first.

The core message we receive from Steiner is that – when introducing mathematical operations to children – we need to describe the process as a whole, from start to end result, with an emphasis on the question what happens to make the change, which is the mathematical process itself. This is different from the open-ended question “we start with this, we do that, so what is the result?”, which later, when the children have experienced the operations of course becomes a strong purpose of doing maths. Before the children are ready for that step, they will reflect on many processes as a whole.

These considerations support what was said about the equal sign as a symbol of balance (see page 4), which encourages us to use a variety of ways of writing sums and a variety of mathematical questions we can ask about each of the operations (e.g. $8 + 3 = \boxed{}$ and $8 + \boxed{} = 11$ and $\boxed{} + 3 = 11$). By practising all these varieties we truly develop the children’s understanding, rather than training them for a response when they see the equal sign at the end of a problem statement.

THE ACTIVE COMPONENT IN MATHEMATICAL OPERATIONS

In additions, subtractions and divisions, the typical sum XXX + XXX = XXX begins with a starting number and ends with the end result, whereas the active component (the moment where a mathematical operation is active) relates to the number in the middle.

In the sum 8 + 4 = 12 the adding of the number 4 is the active component (addend), which changes 8 to 12.

In 15 – 6 = 9 the subtraction of 6 is the active part (subtrahend).

In 21 ÷ 3 = 7 the division by 3 is the active part (divisor).

However, in multiplications the first number is the active part (multiplier). As described in the paragraph “multiples and tables” (page 16), a multiplication “XXX times XXX” should be interpreted as “XXX groups of XXX”, which means that the number of items in each group is what the operation starts with, then the number of groups is defined, which then leads to the outcome of the multiplication. Changing the active element from 1x, 2x, 3x, etc is exactly why we need to practise the tables in the right order, ie. 1 x 4, 2 x 4, 3 x 4, etc.

Further than practising multiplication tables correctly, the implications of finding the multiplier in the first number of a multiplication should be kept conscious whenever we translate a real life situation (e.g. a story sum) into a multiplication. When, for instance, a farmer planted a row of 12 cabbages and Tommy Times decided that he would do that 3 times, the multiplication should be 3 x 12 and not the other way around, to stick to the image of rows of 12. Turning the multiplication around would incorrectly create rows of three!

Verbalising multiplications correctly is mainly important in the early years of doing mathematics, as later on, the commutative aspect of multiplication will allow us to be more flexible with the order of the numbers. In higher primary classes we might say “so we have 25 and now we multiply that by 2 and 25 x 2 = 50” without realising that we actually meant 2 groups of 25.

IMAGINATION IN SUPPORT OF NUMBERS AND OPERATIONS

The use of imaginative elements in teaching mathematics (e.g. stories, blackboard drawings, artistic activities) is one of the key aspects of Waldorf education, as these images relate the content of the learning to the children’s feeling realm, which in turn forms a bridge to the moments when the focus is on the will (e.g. rhythmical activities and games). To make sure that the imaginative elements are indeed a bridge between thinking and will, it is important to choose one’s images and stories in ways that can truly interrelate with the cognitive aspects of the learning, as well as with the practical activities.

Choosing stories and images carefully first of all means that the mathematical content they represent must be true and relevant. Stories can either move too far away from the actual number work and start to lose their connection, especially when they are drawn out too long and contain very little maths content, or the opposite: very short stories that can hardly be called imaginative, and sound like mathematical problems wrapped up in a not very relevant story, because of the principle of teaching through stories.

The common story for the introduction of the four operations (Farmer Plus, Tommy Times, etc.) has been used by many teachers over many years. This already leads to the observation that we need to encourage teachers to find new ideas, instead of repeating the old ones. Each story and image has its pros and cons, and teachers should not shy away from choosing good, different ideas. The creative process on the side of the teacher will certainly find a kind of reflection in the children, that is different from re-using an old concept.

Important to realise is that the stories and images neither teach the children full understanding nor the skills they need. They are a valuable contribution to the learning and the practising of a concept, but the thinking and the will cannot depend on what has been learnt in the feeling realm. Moments of conscious thinking about a concept, discussing how children would tackle a question, doing quiet mental maths, or practising individual written work, are the times when thinking skills are developed. Jumping and clapping tables and bonds, number games and practising instant responses during bean bag activities, are the times when maths is practised at the level of the will. At the stage of these thinking and will activities, an occasional reminder of the stories and images will keep the children connected to the imaginative introduction, will enhances their motivation for the practising, and boosts their confidence towards what they aiming for.

STORY SUMS AND WORD PROBLEMS

Much of what was said above about using imagination, at the stage of introducing mathematical concepts, also applies to numerical questions woven into a story or word problem. This stage of maths is usually practised after children have learnt the necessary calculations as isolated, mathematical operations. Presenting similar questions as a story or word problem then adds the difficulty of needing to decide how the verbal information from the story is to be translated into the mathematical question statement.

As above, these stories and word problems need to be true (which includes ‘believable’) and relevant to the mathematical concept that is being practised. Untrue elements of such stories might relate to incorrect statements (e.g. a mother bought 3 loaves of bread for R 1,50 each – but a loaf of bread costs at least R 8) or unbelievable situations (e.g. a mother carried 250 apples in her bag – but this number of apples will not fit into a bag). Irrelevance occurs for instance when the question would never be asked in real life, as in “3 friends live in the same road, A at no.10, B at no. 14 and C at no. 20. How much less is C’s house number, than A and B together?” Another problem occurs in the description “Farmer A grew 20 cabbages and farmer B 80 cabbages. How many times more cabbages than farmer A did farmer B grow?” (is the answer 4x, as 4x20=80? or 3x, as B grew 60 more, which is 3x20?).

Very careful consideration needs to be given to these stories and word problems. A conscious decision is necessary whether or not to include information that is not relevant for the calculation (e.g. Susan is 9 years old, her brother Paul is 2 years younger and her brother Jack 1 year older than she – how old is Paul?). At some stage, children will of course have to learn to differentiate between relevant and unnecessary information, but the teacher needs to determine which step can be taken at which time. Also the order of the information is important, as this can be similar to, or opposite the order of the numbers in the calculation (e.g. John has R30, Peter has R6 less. How much does Peter have? – vs. – Peter has R6 less than John. John has R30. How much does Peter have?).

Children’s ability to deal with word problems will greatly depend on the amount of mathematical understanding they have been able to develop during the practising of the calculations just with numbers. Having learnt to do calculations with a thorough understanding of the ‘real’ numbers, as opposed to the vertical digit-by-digit techniques (see page 5) will enable them to think about the word problem, to analyse what should happen with the numbers, and to come to a logical conclusion.

Logic in mathematical thinking is often the main reason for being successful in word problems. Knowing what to do with the following problems, for instance, will be impossible without thinking logically.

If it takes 10 farm labourers 6 days to plant 900 young grape vines, how long would it take if the work was done by 30 farm labourers?
If it takes 10 farm labourers 6 days to plant 900 young grape vines, how many grape vines could 30 farm labourers plant in the same time?

In these examples one has to choose between answering to the multiplication (10x3=30) with another multiplication or with a division. In the first example a division is needed: There are 3 times as many labourers, so the work will be done in 1/3 of the time, so 6 ÷ 3 = 2 days. In the second example a multiplication brings the result. There are 3 times as many labourers, so in the same time 3x as much work will be done, so 900 x 3 = 2700 grape vines. Without logical thinking there is no clue to predict which operation is correct and children who have been taught strictly regulated algorithms to come to their answers digit-by-digit will be at a strong disadvantage compared to children who have been encouraged to deal with the ‘real’ numbers and come with their own solutions for various problems.

As in all teaching and learning of mathematics, ongoing practising of word problems gives the best chance that children will develop the insight they need. Many teachers underestimate the amount of practice needed as regards word problems, and should spend more, and more regular time on problems and calculations that are described in words, not merely in numbers and symbols.

BODMAS

The acronym BODMAS refers to the order in which multiple operations in one sum need to be carried out, instead of just doing each step in the order of the sum itself. It is an agreement between mathematicians (not a principle of mathematical truth) to do some operations before others, such as multiplications before additions. The difference becomes clear with an example: 3 + 2 x 4 if calculated along the order of the sum itself would lead to 3 plus 2 is 5, times 4 is 20. The correct approach however is to do the multiplication first, so 2 times 4 is 8, which we add to 3 for the result 11.

The acronym BODMAS stands for:

B – Brackets
O – 2 meanings: Of & Order
D – Division
M – Multiplication
A – Addition
S – Subtraction

Brackets are used to give priority to anything that should NOT be calculated in the BODMAS order. If in the above example the writer of the sum means "add 3 and 2, then multiply the result by 4", brackets are necessary to indicate this, ie. (3 + 2) x 4. Brackets sometimes appear unnecessarily, e.g. 3 + (2 x 4) which is mathematically not incorrect, but should be avoided.

Of & Order are two different interpretations of BODMAS, both of which are not very important, as they in fact sort themselves out when they appear in a sum. The term 'Of' in maths means the same as a multiplication (e.g. 3% of 200 = 3% x 200, and half of 10 = $\frac{1}{2}$ x 10. The 'Of' principle therefore easily fits in with doing the multiplications. The term 'Order' refers to power raising, e.g. $2^3 = 2 \times 2 \times 2$. This hardly leads to questions either, as few people would solve for instance $4 + 2^3$ incorrectly as "4 plus 2 is 6, to the power of 3 is 216". The power raising 2^3 is naturally done first, unless brackets show us $(4 + 2)^3$.

Divisions and Multiplications are together a subgroup in BODMAS which means that divisions and multiplications are dealt with as they appear in the order of the sum. For example 4 x 12 ÷ 3 x 2 is solved as "4 times 12 is 48, divided by 3 is 16, times 2 is 32".

Likewise, Additions and Subtractions are a subgroup in BODMAS which means that additions and subtractions are dealt with as they appear in the order of the sum. For example 6 + 9 – 5 + 2 is solved as "6 plus 9 is 15, minus 5 is 10, plus 2 is 12". It is NOT correct to do all the additions first and then the subtractions, as this would solve the sum incorrectly as 6 + 9 – 7 = 8.

In short, the main purpose of BODMAS is to inform us that any multiplications or divisions in a sum need to be done first, followed by the additions and subtractions, unless brackets force another order of events. Children could easily learn this principle without hearing about the BODMAS acronym, but the teaching of the BODMAS principle to higher primary classes is often well received, so why not?

While practising the application of the BODMAS rules, it is helpful for children to compare the correct result, using BODMAS, with different, incorrect answers obtained by using the order of the operations wrongly. This will make them realise the significance of the rules.

ROUNDING OFF AND ESTIMATION

The ability to estimate, ie. to predict to a certain extent the result of what one could calculate more accurately, is a very important and often undervalued skill in mathematics. This is not guessing (which is too random to be useful) but based on interpreting rounded-off numbers to roughly predict the outcome.

Estimating gives a person the possibility to form a hypothesis before starting the work (e.g. I think it will be around 100), and then to check that the outcome is not completely out of line with the expectation.

Already at the stage of counting items (class 1), estimation can be practised, by asking the child "How many do you think there might be on your table?" before starting to count the exact number. This estimation practises the ability to get a feel for the number, e.g. between 10 and 20, or around 50, or well over a hundred.

When the pupils have moved away from working with counters, they should practise estimating the answer before the accurate calculation. E.g. 35 + 28 will be 60 something, because 30 + 30 is 60 and then there are still a few left.

Estimation goes hand-in-hand with rounding off, but uses this informally, not in the structured sense of "Round off 94 to the nearest ten". Part of learning to estimate is to develop a feeling for what level of rounding off will be acceptable for the purpose of the estimation. In addition and subtraction rounding off all numbers to the same extent will be acceptable, but estimations for multiplications or divisions can be affected by small amounts of rounding off. For instance in the multiplication 10.5 x 24 the rounding off to 10 x 24 gives 240, which is pretty close to the accurate answer 252. However, if the multiplication is 1.5 x 24 rounding off to 1 x 24 gives 24, which is quite different from the accurate 36. Alternatively 2 x 24 would give 48 which is quite far the other way. In the number 1.5 the 0.5 is in fact too important to be rounded off. Rounding off for estimation therefore needs to be done with a feeling for the relative importance of the change, which is a difficult concept, especially for children. Practising it, and comparing the estimated and the accurate answers, is the only way to build this skill.

The structured process of rounding off, according to instructions (e.g. round off to the nearest whole number, the nearest ten, the nearest thousand) has its own place, and needs to be practised. The rule that one rounds off to the nearest number of the range indicated is quite easy, apart from the exact middle, which then rounds off upwards. It will help children to define the two numbers from the range that are neighbours and to state how the rule will be applied. Some examples:

to the nearest ten	to the nearest whole number
14 rounds off to 10, as it's closer than 20	34.32 rounds off to 34, closer than 35
26 rounds off to 30, as it's closer than 20	165.623 rounds off to 166, closer than 165
35 rounds off to 40, halfway 30 / 40	403.50 rounds off to 404, halfway 403 / 404
64.9 rounds off to 60, as it's closer than 70	45.495 rounds off to 45, closer than 46

CLASSES 1 to 3 – NUMBERS AND OPERATIONS

Through the various paragraphs in this booklet, an overview has been given of the principles through which the children can develop a thorough understanding of numbers and the four processes. A strong plea is included to encourage working with 'real' numbers, for which it is necessary to avoid the traditional digit-by-digit algorithms throughout the classes 1 – 3. For details please refer to the section on expanded calculations from page 5.

In class 1, after the introduction of the numbers and their qualities, the operations will naturally happen with the real numbers in the sums, as these numbers are still quite simple. Refer to paragraphs about the three levels of learning operations (page 10), the three levels of visual aids (page 12), the three levels of practising (page 13) and about working from the whole to the parts (page 21) for important teaching approaches at this level. Learning bonds and multiples towards the level of instant responses (page 14) will create a solid basis for the work in the next class.

In class 2, the focus will be on the children's ability to do decomposition to enable them to do the operations with more complex numbers (e.g. tens and units). The principles of the expanded calculation, described from page 5, will guide the teacher both in the written work the children do, as well as in mental maths, because expanded calculation is not limited to the notation methods. Spending much time on practising maths skills, both during activities (the multiplication tables are starting, various bonds and 'easy' additions, see page 14) and mental and written calculations, will be needed to develop and practise the skills the children need.

In class 3, the bigger numbers will make the calculations much more challenging. The expanded calculations can handle all these situations efficiently, as the paragraphs from page 6 describe. At the level of instant responses (see page 14) the children will now need to show individual competence in what has been practised in multiplication tables, bonds and other 'easy' maths work. At the level of more complex mental and written maths, the children will need to spend time on maths every day, in order to build up sufficient skills, speed and confidence. Class 3 is really the time to make sure that all the basic maths requirements are in place, to pave the way for the fractions, decimals and percentages in consecutive years.

Although the theme-based maths main lessons such as measuring and money are very enjoyable and meaningful for the children at the level of their imagination, the drawback of focussing on these themes in class 3 is that the practising of skills in the normal operations might receive too little attention. It is therefore important to allocate running lessons, in addition to the theme-based main lessons, in which the maths skills are practised and refined.

CLASS 4

FRACTIONS

The way in which fractions are introduced to class 4 children will be an important factor determining how well they will cope with fractions at later stages of learning maths. The basic concepts need to be taught slowly and thoroughly, and with a continuous connection to the 'real' thing represented by the fraction. The 3 learning steps i) concrete, ii) diagrams and iii) no visual as described on page 12 are especially relevant for the first main lessons teaching the fractions.

The first fractions main lesson

At first, the fractions should be represented by hands-on materials, e.g. fragments of circles cut into halves, thirds, quarters, etc. With these fragments the children can do their first fraction puzzles, to find out that for instance $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$ (one whole). In this way children learn about the different families of fractions, which together have the ability to form a whole. One family consists of the halves, quarters, eighths, sixteenths, etc. The next family has the thirds, sixths, ninths, and twelfths. Then there is the family of fifths, tenths and twentieths. Other fraction families are the odd ones like sevenths, elevenths and thirteenths. Each member of a fraction family has much in common with its relatives, and make a puzzle fit well.

The notation of one number over another number is completely new for the children, and needs to be treated as new for quite a while. Teachers often forget that they themselves have know the fractions for a long time, whereas children need a process of familiarising themselves with the notation and its meaning. To help this process, it is strongly recommended that while we work with the different fractions (denominators) we keep the 1 on top (numerator) <u>for the whole of the first fractions main lesson</u>. This will give children the opportunity to focus on the denominator and sense the family quality of the fraction they deal with, without being distracted by a changing numerator. This qualitative approach instils a strong connection to each fraction family and avoids confusion between the roles of the number on top and the one below.

As in the example above, one can do endless sums with single halves, thirds, quarters, etc. making one whole, two or more wholes, and soon also a half, a whole and a half, etc. Each time the written reflection on the work will be done with the numerator 1, for instance: $\frac{1}{3} + \frac{1}{6} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1\frac{1}{2}$. After a while, the writing of a whole series of the same fractions will become quite tedious, e.g. $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$. At this point the teacher can reveal to the children that this repetitive work can be reduced by writing $4 \times \frac{1}{8} = \frac{1}{2}$ which is a better way of avoiding the repetition (at this stage) than jumping to the introduction of different numerators as in $\frac{4}{8}$. By continuing using the numerator 1, now combined with a multiplier, the children's connection to the quality of the fraction will remain optimal. This way, the children can learn to add, subtract and multiply fractions within the first fractions main lesson, keeping the one on top all the time. Please note the difference between $4 \times \frac{1}{2}$ which means four times the fragment representing the half, and $\frac{1}{2} \times 4$ which means half of four. As described in the paragraph about commutation (page 18) these differences are of great importance when we speak to children about operations, especially regarding fractions. Both ways of using fractions need to be practised. The first version with the circle fragments on the children's desks, the latter version during rhythmic time, when the class investigates how it can form halves, thirds and quarters of its total number of children.

In addition to circle fragments, the children can investigate fractions from other shapes, such as rectangles and triangles, but the great advantage of the circle is that each fraction automatically relates to the complete circle from which it was cut. After cutting fragments of other shapes, the fraction piece will always need to be compared with the original shape to determine its relationship to the original. The whole is not automatically recognised in the shape of the fraction.

The second fractions main lesson

After a thorough introduction to the concept of fractions, the children's level of understanding is ready for further steps. In the 3 steps of i) concrete, ii) diagrams and iii) no visuals the concrete step will now make place for diagrams, although the circle fragments will remain available for those who need to be reminded.

Now it is the right time to introduce other nominators than the number one, and the 'true' work with fractions is beginning.

One of the key concepts in fractions is of course the way in which fractions of the same total value can be written in different ways, for instance $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$. This process of conversion from one family member to the other needs to be practised thoroughly, as this is the basis for many future operations with the fractions. The process of reducing the numbers (nominator and denominator) is of course known as simplification, and it is good to introduce another term for the opposite process, such 'complexifying' or 'complicating', so that children in one word understand what it is that they are expected to do.

Finding all the equivalents of various fractions (up to about 12 terms each) will first of all provide the children with much needed practice.

$\frac{1}{3}=\frac{2}{6}=\frac{3}{9}=\frac{4}{12}=\frac{5}{15}=\frac{6}{18}=\frac{7}{21}=\frac{8}{24}=\frac{9}{27}=\frac{10}{30}=\frac{11}{33}=\frac{12}{36}$
$\frac{2}{5}=\frac{4}{10}=\frac{6}{15}=\frac{8}{20}=\frac{10}{25}=\frac{12}{30}=\frac{14}{35}=\frac{16}{40}=\frac{18}{45}=\frac{20}{50}=\frac{22}{55}=\frac{24}{60}$

However, from the above exercises the patterns of numbers easily reveal to the children how fractions work. They will happily recognise the multiplication patterns and start getting an understanding of what happens in the complicating or simplifying processes that keeps the total value of the fraction unchanged. This realisation often happens especially when reaching $\frac{1}{3}=\frac{10}{30}$ which clearly shows that both numerator and denominator have been multiplied by ten. This key principle of fractions has thus been discovered by the children themselves, rather than been taught by the teacher as a concept they must learn and practise.

Practising the conversions will of course still need to be done thoroughly, as otherwise the cognitive discovery made by the children is left to be forgotten again. Therefore, after producing many series of fractions as shown above, the children can start working out challenges like $\frac{1}{5}=\frac{???}{20}$ and $\frac{21}{24}=\frac{???}{8}$. The latter conversion (simplifying) is making use of division, as numerator and denominator are now divided by the same number to find the simplified equivalent.

Mixed fractions also need to be investigated, and children need to learn that in $\frac{13}{10}$ there is a whole number in the form of $\frac{10}{10}$ which brings the total result to $\frac{10}{10}+\frac{3}{10}=1\,\frac{3}{10}$.

With the above knowledge of mixed and proper fractions, simplifying and complexifying, children can learn to tackle the four operations with fractions.

Addition and subtraction of fractions can only be done if the fractions have the same denominator. At this stage, children should either write out the series of conversions (complicating) for each of the two fractions, until they find a common denominator, or use multiplications to get to the same results without writing the whole series. What one is looking for here is the Lowest Common Multiple between the denominators. At a later stage (e.g. class 6) children could learn to find the LCM through factors, but now just going through the multiples is the preferred approach. Once the common fraction has been found, an addition is straightforward, but a subtraction can be hindered by the need to convert a whole number to the fraction, see the examples below.

$12\,\frac{2}{3}+14\,\frac{5}{8}=12\,\frac{16}{24}+14\,\frac{15}{24}=26\,\frac{31}{24}=26+\frac{24}{24}+\frac{7}{24}=27\,\frac{7}{24}$ in this example the 31 twenty-fourths are top-heavy and produce another whole number adding to 26
$32\,\frac{2}{5}-8\,\frac{2}{3}=32\,\frac{6}{15}-8\,\frac{10}{15}=24\,\frac{6}{15}-\frac{10}{15}=23+\frac{15}{15}+\frac{6}{15}-\frac{10}{15}=23+\frac{21}{15}-\frac{10}{15}=23\,\frac{11}{15}$ in this example there are not enough fifteenths to take away 10, so one whole out of 24 turns to fifteenths.

Doing multiplications between fractions is a very different process, and quite a difficult one for children to understand, as the result of a multiplication between two fractions is likely to be smaller than any of the two numbers that started the question. To give children a key to understand a multiplication like $\frac{2}{5}\times\frac{3}{4}$ it is a good idea to revisit a much simpler multiplication of the same kind, such as $\frac{1}{2}\times\frac{1}{4}$. Half of a quarter is still a concept that children can imagine, and it can also be visualised with the circle fragments or with a diagram. The result, one-eighth, is clearly smaller than both a half and a quarter were beforehand. The multiplication of the 2 and the 4 already becomes apparent. Now if there wasn't one quarter but three, we would end up with three of those eighths, so $\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$. The principle that when multiplying fractions the top and the bottom numbers are multiplied horizontally becomes visible. Due to this principle, it is necessary to avoid mixed fractions, which means that, contrary to additions and subtractions, we now need to convert the whole numbers into fractions before we can do the multiplication. Although mathematically one could in fact expand whole numbers and fractions and get the right answer, this becomes quite a laborious process and children are unlikely to benefit from expanding these multiplications.

$\frac{2}{5} \times \frac{3}{4} = \frac{6}{20} = \frac{3}{10}$
$1\frac{2}{3} \times 5\frac{1}{2} = \frac{5}{3} \times \frac{11}{2} = \frac{55}{6} = 9\frac{1}{6}$

A division between two fractions is an almost impossible concept to grasp, and as a result, the trick of how to work out such a problem (through a multiplication with the opposite fraction) is likely to be the only thing children will learn about divisions of fractions. However, there are various ways to try and give meaning to what it is to divide by a fraction. Of course, the concept of division is normally based on a number of recipients of a certain quantity, and the question how many items each recipient receives. This concept obviously does not apply to a division by a fraction. Instead, a simple division by a fraction such as $4 \div \frac{1}{2}$ can be converted to the question "How many halves can fit in 4?" The answer will be 8, which is a glimpse of understanding towards the difficult concept that the answer of a division by a fraction is likely to be much <u>bigger</u> than any of the numbers involved in the problem. We are now also able to show children that the operation, taken back to front, is in fact the multiplication $8 \times \frac{1}{2} = 4$ in the same way as a normal division such as $35 \div 7 = 5$ turns into $5 \times 7 = 35$.

Divisions between fractions can in fact be done exactly the same way as multiplications, ie. dividing the numerators horizontally and doing the same with the denominators. This often results in fractions within fractions (which is mathematically correct but not at the level of primary learning) unless we choose suitable combinations of numbers, see the examples below.

$\frac{28}{6} \div \frac{1}{2} = \frac{28}{3}$ (which is $\frac{28 \div 1}{6 \div 2}$)	$\frac{28}{6} \div \frac{7}{2} = \frac{4}{3}$ (which is $\frac{28 \div 7}{6 \div 2}$)	but $\frac{2}{5} \div \frac{3}{4} = \dfrac{\frac{2}{3}}{\frac{5}{4}}$

The first two examples above can show the children that the principle of dividing horizontally is the same as multiplying, but as soon as non-compatible numbers come in (the third example) it is clear that something else is needed. There is the possibility of converting both fractions to a common denominator and then converting the fractions to whole numbers, see the following example, but few class 4 teachers may want to try that with their class.

$\frac{2}{5} \div \frac{3}{4} = \frac{8}{20} \div \frac{15}{20} = 20 \times \frac{8}{20} \div 20 \times \frac{15}{20} = 8 \div 15 = \frac{8}{15}$

Therefore, despite all the explanations that can be tried, the way of handling divisions between fractions will end up using the trick to turn the second fraction upside down and to turn the division into a multiplication.

$\frac{2}{5} \div \frac{3}{4} = \frac{2}{5} \times \frac{4}{3} = \frac{8}{15}$

Hopefully this will be done with at least some understanding of division by a fraction based on the above attempts. This level of understanding (probably very little in class 4) can be expanded by revisiting the division-by-fraction principles again in later years. It is important to let the children be aware that they have been taught a trick (the multiplication) to temporarily handle a concept that is complicated, but still a real mathematical principle. At least the teacher should have familiarised him/herself with the above explanations, which will help finding the right words when speaking about the multiplication trick.

With the possible exception for division between fractions, the aim for the work with the children is to develop their thinking skills, and to understand the mathematical concepts, rather than having learnt rules and tricks. Avoiding tricks, which is one of the key messages of this booklet, means working with true reasoning skills, based on the numbers that the question presents.

Tricks such as <u>cancelling</u> in fraction problems are very tricky in the sense that children may at first apply the learnt trick of cancelling correctly, but at a later stage get confused as the rules are in fact quite complicated. Can one cancel when doing an addition or subtraction? Or only in multiplications? Can one cancel horizontally, vertically or only diagonally? And as with the digit-by-digit vertical algorithms in whole numbers, the cancelling in fractions removes the children from the process that should become visible in how the real fractions are handled throughout the operation. Therefore, a strong plea to AVOID the teaching of cancelling at least until class 6 or 7. Handle the real fractions and simplify at the end!

CLASS 5

DECIMAL FRACTIONS

When in class 5 the decimal fractions are introduced, the children will heavily depend on their knowledge of the normal fractions. It is essential that children have a thorough knowledge of the normal fractions before moving on to the decimal fractions.

A good understanding of the normal fractions connects the children's learning of decimals with what they have learnt before. It is not impossible to learn about decimal fractions as an isolated area of numbers, but such learning will certainly remain very superficial, without much potential to grow to full understanding and confidence.

Understanding decimal fractions using the normal fractions

As preparation for the introduction to decimal fractions, the normal fractions should be revised with a special emphasis on the fractions in the family of tenths / hundredths / thousandths, etc. The conversion from wholes to tenths, from tenths to hundredths, etc. is especially important.

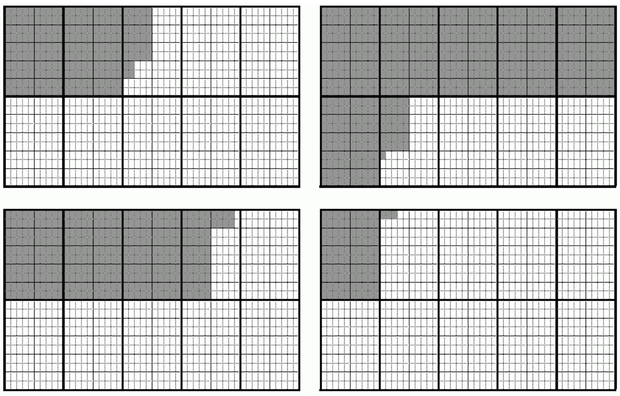
After this revision, the decimal notation can be introduced as a new way of writing the well-known tenths, hundredths and thousandths. Children have hopefully built up a good understanding of place value in whole numbers (which consist of units, tens, hundreds, etc). Refer to the paragraph about place value, and the plea to avoid a digit-by-digit approach, on page 19. This understanding of place value is now converted to the understanding that after the decimal comma (or point, especially on computers) the place values continue with the tenths, the hundredths, the thousandths, and further into smaller and smaller fractions.

The most common mistake in decimal fractions is that a fraction like $\frac{25}{100}$ is written as 0,025. This is in fact completely logical if one considers: "There are no whole numbers, write zero, then the decimal comma, then there are no tenths, write zero, then there are 25 hundredths, write 25. This is incorrect of course, but for children to understand why it should be 0,25 takes a significant learning curve.

In order to prevent the above mistake, and to build in children an real understanding of decimal fractions, the teaching of decimals should focus on the principle that each decimal place value has only space for a one-digit number (as is the case in the place values for whole numbers). The first decimal fraction can have up to 9 tenths, the next up to 9 hundredths, etc.

In the first weeks of learning about decimals, each decimal number should be read out aloud in tenths, hundredths, thousandths, etc. The understanding of this principle can be supported with a grid showing division of a whole rectangle in ten tenths (rectangles of a similar length:width ratio), each tenth subdivided in hundredths and each hundredth subdivided in thousandths. Children can colour in the grid to show various decimal fractions up to thousandths. Photocopiable sheets with this grid can be found after page 36.

The following diagram shows the numbers 0,234 (top left), 0,631 (top right), 0,358 (bottom left) and 0,103 (bottom right).



The first calculations with decimal fractions

True to the principle that children should explore each new mathematical area from a well-known basis, the first calculations involving decimal fractions are solved by using the normal fractions that the children have already understood well. In the following example the decimal fractions are converted to normal ones, then the calculation is made, after which the result is written as a decimal fraction again. This provides many opportunities for the children to familiarise themselves with the fact that a fraction like $\frac{25}{100}$ is written as 0,25 due to the combination of $\frac{20}{100} + \frac{5}{100} = \frac{2}{10} + \frac{5}{100} = 0{,}25$.

12,375 + 14,916	$= 12 + \frac{3}{10} + \frac{7}{100} + \frac{5}{1000} + 14 + \frac{9}{10} + \frac{1}{100} + \frac{6}{1000}$ $= 26 + \frac{12}{10} + \frac{8}{100} + \frac{11}{1000}$ (wholes and same fractions are added) $= 26 + \frac{10}{10} + \frac{2}{10} + \frac{8}{100} + \frac{10}{1000} + \frac{1}{1000}$ (remove what is more than one-digit) $= 26 + 1 + \frac{2}{10} + \frac{8}{100} + \frac{1}{100} + \frac{1}{1000}$ (convert removed values to next level) $= 27 + \frac{2}{10} + \frac{9}{100} + \frac{1}{1000}$ (final adding of same fractions) = 27,291
4,5 x 12,7	$= 4 \times 12 + 4 \times \frac{7}{10} + \frac{5}{10} \times 12 + \frac{5}{10} \times \frac{7}{10}$ (similar to expanded multiplying on page 5) $= 48 + \frac{28}{10} + \frac{60}{10} + \frac{35}{100}$ (do the four multiplications) $= 48 + \frac{20}{10} + \frac{8}{10} + 6 + \frac{30}{100} + \frac{5}{100}$ (what is more than one-digit) $= 48 + 2 + \frac{8}{10} + 6 + \frac{3}{10} + \frac{5}{100}$ (convert removed values to next level) $= 56 + \frac{11}{10} + \frac{5}{100}$ (the tenths are again more than 9, move 10 into wholes) $= 57 + \frac{1}{10} + \frac{5}{100}$ = 57,15

As always, practice is the best tool for working towards understanding of the mathematical principles. Once the children have mastered the above calculation by using the normal fractions, the same logic can be applied when calculating with the decimals themselves, without the conversion. In the following table, the above calculations are repeated, now with decimals only.

Note, that these first decimal calculations are using the children's years of experience with the expanded calculations. In the past, teachers would have just shown the children the digit-by-digit algorithms, explained the role of the decimal comma, and from then on children would have disconnected from trying to understand decimal fractions any further. Nowadays, we know that children need to work with the real numbers much longer, and only should be given the quick tricks as a handy tool once they have a good understanding of what actually happens in these calculations. The in-between step of vertical calculations with real numbers, also including decimal fractions, is described on page 9.

12,375 + 14,916	= 12 + 0,3 + 0,07 + 0,005 + 14 + 0,9 + 0,01 + 0,006 (full decomposition) = 26 + 1,2 + 0,08 + 0,011 (children now understand how extra tenths become wholes) = 27,28 + 0,011 (first done the easy steps) = 27,291 (then the final bits)
4,5 x 12,7	= 4 x 12 + 4 x 0,7 + 0,5 x 12 + 0,5 x 0,7 (similar to expanded multiplying on page 5) = 48 + 2,8 + 6 + 0,35 ($\frac{1}{2}$ of 12 is 6, and $\frac{1}{2}$ of $\frac{7}{10} = \frac{1}{2}$ of $\frac{70}{100} = \frac{35}{100}$, <u>not 3 $\frac{1}{2}$ of course</u>) = 56,8 + 0,35 (first done the easy steps) = 57,15 (then the final bits)

CLASS 6

PERCENTAGES

In class 6 the introduction to percentages is in fact a minor step, compared to the fractions in class 4 and the decimals in class 5.

The most important aspect of teaching percentages is to train the children to ask themselves the first question: "What is one percent of this number?", before moving into calculating the requested percentage. The important principle that one percent is always one hundredth ($\frac{1}{100}$) of the original number is easily understood by children, and so is the conclusion that dividing the original number by 100 will provide us with the one percent that we would like to know. After this, the requested percentage (e.g. 14% VAT) can be found by doing the relevant multiplication.

Many teachers choose to teach percentages in the form of $\frac{\text{perc}}{100} \times \frac{\text{original}}{1}$, after which some cancelling happens, before the fractions are multiplied. Although this might be a useful approach to know at a much later stage, the early introduction of this method unfortunately causes many pupils to lose their connection to percentages at the stage where they should really be practising percentages as a basic concept. The standard first question "What is one percent of the number?" assist greatly in approaching percentages at the correct level of understanding.

Having been introduced to percentages like this, and having had plenty of practice to turn this understanding into a skill, children should develop an understanding of the connection between certain percentages and the equivalent fraction. The most prominent ones are listed below.

100% = the whole in other words 1	$50\% = \frac{1}{2}$	$25\% = \frac{1}{4}$	$75\% = \frac{3}{4}$
$20\% = \frac{1}{5}$	$40\% = \frac{2}{5}$	$60\% = \frac{3}{5}$	$80\% = \frac{4}{5}$
$10\% = \frac{1}{10}$	$30\% = \frac{3}{10}$	$70\% = \frac{7}{10}$	$90\% = \frac{9}{10}$
$5\% = \frac{1}{20}$	$15\% = \frac{3}{20}$	$33\frac{1}{3}\% = \frac{1}{3}$	$66\frac{2}{3}\% = \frac{2}{3}$

Most of these conversions between a percentage and a fraction can be calculated by writing the percentage as a fraction out of a hundred, and then simplifying, e.g. $35\% = \frac{35}{100} = \frac{7}{20}$. The third, sixths and other fractions that cannot be written as a decimal fraction (with a limited number of places behind the comma) are of course exceptions and children will learn that it is acceptable to keep a fraction within the percentage. In primary education children are not yet taught that it is mathematically correct to do the same in a normal fraction, as in $33\frac{1}{3}\% = \frac{33\frac{1}{3}}{100}$.

Conversions between normal fractions and decimal fractions (including percentages) can also be found by using a long division, as long as one is familiar with passing the decimal comma in these long divisions.

$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{2}{7}$
0,125 8 | 1,000 1 0 20 16 40 40 0	0,33333 3 | 1,00000 9 10 9 10 9 10	0,14285714 7 | 1,00000000 7 30 28 20 14 60	0,28571428 7 | 2,00000000 1 4 60 56 40 35 50

	9 10 9 1	56 40 35 50 49 10 7 30 28 2	49 10 7 30 28 20 14 60 56 4

CLASS 7

NEGATIVE NUMBERS

As the class 4 child needs a slow and thorough way of entering into the new world of fractions, the same applies to the class 7 child entering the new world of negative numbers. Also here the teaching of tricks (double-minus is plus) stands in the way of teaching a full understanding of what negative numbers are about.

There are various practical uses of negative numbers that the class 7 have already heard about, which can be revisited as part of the introduction to negative numbers. Temperatures above and below zero, altitudes above and below sea level are some examples. Although these examples are useful to a certain extent, the problem with these two is that a negative temperature and a negative altitude still represent a temperature or an altitude, not the absence of such. One of the clearest examples therefore is having possessions (positive numbers), nothing at all (zero), or debts (negative numbers).

Number lines (see page 20) are an excellent tool for working with the negative numbers. Here the reference to altitudes or temperatures (for a number line drawn vertically) will be useful. Realising that the number 0 is no longer the lowest possible value, but that subtracting a larger number from a smaller number is now possible, can easily be shown in such a diagram. If the temperature was first 5 degrees during the day, and at night the temperature dropped by 8 degrees, the diagram easily shows that the result is 3 below zero, which is –3. When verbalising such a negative number, the terms ‘negative 3’, ‘minus 3’ and ‘3 below zero’ should all be used from time to time. If it is about money or possessions, include ‘a debt of 3’.

Stepping up and down a number line, drawn on the floor, is a good way to learn about negative numbers through movement. Agreeing on a few home-made guidelines is all that is needed to be able to visualise all children need to know about negative numbers. These guidelines could for example be:

1. An addition is done facing UP the number line, and a subtraction is done facing DOWN. For example the addition 4 + 5: a child stands on number 4 facing UP, takes 5 steps forward and reaches 9. The subtraction 13 – 7: a child stands on number 13 facing DOWN, takes 7 steps forward and reaches 6.
2. A positive number is done walking forward and a negative number backwards, whichever way the child is facing.

With these guidelines, adding and subtracting negative numbers, even starting from a negative number, becomes visible as follows:

4 – 7	start on 4, face down the number line, take 7 steps forward, reach $\boxed{-3}$
6 + $\boxed{-5}$	start on 6, face up the number line, take 5 steps backwards, reach 1
$\boxed{-2}$ + 8	start on $\boxed{-2}$, face up the number line, take 8 steps forward, reach 6
$\boxed{-2}$ + $\boxed{-10}$	start on $\boxed{-2}$, face up the number line, take 10 steps backwards, reach $\boxed{-12}$
3 – $\boxed{-5}$	start on 3, face down the number line, take 5 steps backwards, reach 8

In the written versions of the calculations with negative numbers it is, at this introductory stage, important to find a clear way of distinguishing between the – symbol indicating a subtraction and the same symbol indicating a negative number. The boxes around negative numbers in the above examples serve this purpose, but using a different colour for the $\boxed{-5}$ will also emphasize that the symbol and the number belong together.

By verbalising the operations with negative numbers correctly, a teacher can also contribute much to the understanding children develop about these numbers. Again, the principle of a debt is the most helpful reference. In the attempts to bring understanding for the examples already used, it is important for the children to realise that someone can own possessions and at the same time have outstanding debts, resulting in a total balance. This balance can be positive or negative. Without this principle, the last of the above questions is hard to explain. The following sentences verbalise the questions done as number line steps above.

4 – 7	I had 4, I gave away 7 (for which I had to borrow), which resulted in a debt of 3
6 + $\boxed{-5}$	I had 6, but I also had a debt of 5, so that in total I only possessed 1
$\boxed{-2}$ + 8	I had a debt of 2, I then received 8, so after paying off my debt I had 6
$\boxed{-2}$ + $\boxed{-10}$	I had a debt of 2, I then added more debt (10), resulting in a debt of 12.
3 – $\boxed{-5}$	My total possessions were 3 (probably the balance between owning things and certain debts), I then reduced my debts by 5, after which my balance was 5 more than it was

Multiplications involving negative numbers are also best understood by referring to possessions and debts. As long as there is only one negative number, the word debt can be used both before and after the multiplication sign. For example 3 x $\boxed{-7}$ is three times a debt of 7, which is a debt of 21, and $\boxed{-2}$ x 8 is two shortages of a group of 8 each, which is a shortage of 16. The term 'shortage' will be needed when verbalising the double negatives, such as $\boxed{-4}$ x $\boxed{-5}$, which becomes 4 shortages of a debt of 5. A shortage of debts is a reasonably convincing way to bring across that a double negative in multiplications produces a positive number.

Divisions involving negative numbers are understandable as long as the divider is still positive. $\boxed{-10}$ ÷ 5 can refer to a debt of 10 divided over 5 people, so they all carried a debt of 2. Dividing by a negative number, however, is as difficult to understand as dividing by a fraction (see page 26). However, by now the class 7 children will have understood enough of the negative numbers to make sense of this situation, for instance through the question 'how many of the (negative) divider would fit in the first number'? If the first number is positive, then the answer will need to be negative.

Investigating number patterns, from the positive down into the negative, is often an excellent activity to support children's understanding of what they are learning. In the following examples one of the components is reduced by 1 each step, going through zero into the negative. The patterns become evident.

2 + 2 = 4 2 + 1 = 3 2 + 0 = 2 2 + $\boxed{-1}$ = 1 2 + $\boxed{-2}$ = 0 2 + $\boxed{-3}$ = $\boxed{-1}$ 2 + $\boxed{-4}$ = $\boxed{-2}$	2 – 2 = 0 2 – 1 = 1 2 – 0 = 2 2 – $\boxed{-1}$ = 3 2 – $\boxed{-2}$ = 4 2 – $\boxed{-3}$ = 5 2 – $\boxed{-4}$ = 6	2 x 3 = 6 1 x 3 = 3 0 x 3 = 0 $\boxed{-1}$ x 3 = $\boxed{-3}$ $\boxed{-2}$ x 3 = $\boxed{-6}$ $\boxed{-3}$ x 3 = $\boxed{-9}$ $\boxed{-4}$ x 3 = $\boxed{-12}$	2 x $\boxed{-4}$ = $\boxed{-8}$ 1 x $\boxed{-4}$ = $\boxed{-4}$ 0 x $\boxed{-4}$ = 0 $\boxed{-1}$ x $\boxed{-4}$ = 4 $\boxed{-2}$ x $\boxed{-4}$ = 8 $\boxed{-3}$ x $\boxed{-4}$ = 12 $\boxed{-4}$ x $\boxed{-4}$ = 16	6 ÷ 3 = 2 3 ÷ 3 = 1 0 ÷ 3 = 0 $\boxed{-3}$ ÷ 3 = $\boxed{-1}$ $\boxed{-6}$ ÷ 3 = $\boxed{-2}$ $\boxed{-9}$ ÷ 3 = $\boxed{-3}$ $\boxed{-12}$ ÷ 3 = $\boxed{-4}$	8 ÷ $\boxed{-4}$ = $\boxed{-2}$ 4 ÷ $\boxed{-4}$ = $\boxed{-1}$ 0 ÷ $\boxed{-4}$ = 0 $\boxed{-4}$ ÷ $\boxed{-4}$ = 1 $\boxed{-8}$ ÷ $\boxed{-4}$ = 2 $\boxed{-12}$ ÷ $\boxed{-4}$ = 3 $\boxed{-16}$ ÷ $\boxed{-4}$ = 4

RATIO AND PROPORTION

The topic of dealing with ratio and proportion is very closely related to what children have learnt about fractions. Teaching this is therefore an excellent moment to revisit the work with non-decimal fractions, and to inspire the children for the practical use of knowing about fractions.

Ratio is preferably written with a colon : to distinguish it from normal fractions, although some people do regard 3 : 4 and $\frac{3}{4}$ both as acceptable ways of writing down ratio.

The preference for using the colon, especially at the time children are introduced to ratio, relates to the big difference between a ratio and a fraction in terms of its language and meaning. Ratio compares two or more elements, for instance a number of boys and a number of girls. If the number of boys relates to the number of girls as 3 : 4 this means that the total number of children is 7 (or a multiple of it) and that for each 3 boys there are 4 girls. Which element one starts with is not important, as the same example could be transferred to the number of girls relating to the number of boys in the ratio 4 : 3. In a ratio, the placing of the elements (left/right) is therefore flexible. Once this has been chosen, the choice needs to be adhered to, of course.

In a fraction like $\frac{3}{4}$, however, there is no connection between the 3, the 4 and the number 7 (the addition of numerator and denominator) never plays the important role that it does in ratio. The connection between the ratio 3 : 4 and the fraction $\frac{3}{4}$ is that the first element (3) is $\frac{3}{4}$ the amount of the second element (4). This is an important discovery for children starting to work with ratio, but for them to understand ratio as a concept, it is important to keep writing it with the colon.

At the stage of introducing ratio, it is essential to relate each element within the ratio to the total, so that in the above example the emphasis is on the understanding that 3 out of 7 children are boys and 4 out of 7 are girls. This prevents the incorrect conclusion that three quarters of the number of children must be boys – a conclusion easily made when ratios are written as fractions.

Ratios are simplified, or made more complex, in the same way as a fraction (by multiplying or dividing with the same number) for instance 4 : 5 = 4 [x3] : 5 [x3] = 12 : 15 and 30 : 35 = 30 [÷5] : 35 [÷5] = 6 : 7.

As in the case of fractions, the simplest way of writing the ratio is the preferred final result and if the ratio contains fractions itself (which is allowable) the process of simplifying needs to get rid of the fractions, for instance in $\frac{1}{2}$: 3 = $\frac{1}{2}$ [x2] : 3 [x2] = 1 : 6.

Ratios comparing different measurement units need to be converted to the same measurement unit in order to produce the simplest and cleanest ratio representation. For instance 500 gram : 4 kg = 0,5 kg : 4 kg = 1 kg : 8 kg, after which the kg unit can be omitted to define the ratio as 1 : 8.

Comparisons between the different elements within the ratio were already mentioned above, as in the ratio 3 : 4 the first element is $\frac{3}{4}$ of the second element. After this easy step, the more challenging step can be taken towards the conclusion that the <u>second</u> element is $\frac{4}{3}$ of the <u>first</u> element. To make this visible, it helps to avoid working with the 3 and the 4 as amounts, but to use a set of multiples such as 6 and 8. This utilises the principle of using different numbers as described on page 21. With these numbers it is easy to see that indeed the first number is $\frac{3}{4}$ of 8 = 3 x $\frac{1}{4}$ of 8 = 3 x 2 = 6. The other way around we see that the second number is $\frac{4}{3}$ of 6 = 4 x $\frac{1}{3}$ of 6 = 4 x 2 = 8.

Ratios can also show how more than two numbers relate to each other, for instance 2 : 3 : 4. In this case the total is 9 (or 9 multiplied by something) and the elements are $\frac{2}{9}$ and $\frac{3}{9}$ and $\frac{4}{9}$ of the total.

Dealing with proportion is a regular event in solving word problems, such as “If 2 litres of milk cost R 15 what is the price of 6 litres of milk?”. This problem can of course be solved by finding the price of 1 litre first and then multiplying by 6. The ratio method will ask: “How can we convert 2 litres : R 15 to 6 litres : R …. ?” For this, we apply the above method for making ratios more complex, ie. 2 litres [x3] : R 15 [x3] = 6 litres : R 45. Using fractions for calculating proportions leads to $\frac{2}{15} = \frac{6}{???}$ and again, [x3] top and bottom gives us the solution.

As for all mathematical topics, the need is emphasized to work with ratio from an approach of <u>understanding</u> and logical reasoning, rather than applying tricks (such as the old way of calculating ratio with lots of cancelling in fractions).

ALGEBRA

If children’s mathematical thinking has developed well during their primary years, the step towards algebra is not at all a challenging one, but rather a confirmation of what they already understand about working with numbers. Especially ongoing variation of question formats, as described on page 21, will have prepared the children well for this step. In fact, the only change will be that instead of a box where the unknown number is meant to go, we now write a letter of the alphabet, for instance an x. The question 7 – $\boxed{}$ = 3 is now written as 7 – *x* = 3 and we ask ourselves how we can find *x*.

At first, children are merely asked to think creatively about the questions, and to come with an answer for the unknown variable. Step by step, the possibilities of starting to manipulate the algebraic question are explored. It is very important, as mentioned throughout this booklet, to let children take these steps with a thorough understanding of what is being done, instead of teaching them tricks that will bring answers if applied correctly (but not understood).

Working from the whole to the parts is happening when we ask the children to solve the problem first (by creative thinking) and then to analyse what we could have done to find the answer. For instance in the question $x + 5 = 12$ we can easily find that x must be 7. But now, if the numbers had not been so easy, what could we have done to find a solution still? Can the question be re-shuffled and re-written to reach the desired ending format $x = \ldots$? What hinders us in this example is the + 5 to the left of the equal sign. Remembering the equal sign as an indicator of balance (see page 14) and that it is possible to make changes as long as we do the same on both sides of the equal sign, we plan to get rid of the adding of 5 by subtracting 5, which then also needs to be done on the right hand side. We thus solve the question (to which we already know the answer) as follows:

$x + 5 = 12$	
$x + 5 – 5 = 12 – 5$	what we change on the left of the equal sign must also be done on the right
$x = 12 – 5$	and now the + 5 has been eliminated
$x = 7$	

The above example shows important steps towards the child's understanding of what is happening. It would be detrimental at this stage to teach the child to move the + 5 to the right and turn it to – 5. This kind of trick prevents the development of an inner connection with the real mathematical work that is happening.

Once the children have developed a good understanding of how to manipulate with unwanted additions and subtractions, the same can be applied to multiplications and divisions. What we do to get rid of something on the left, must also be done on the right, to keep the balance represented by the equal sign.

$3x = 36$	it is easy for children to learn that the multiplication sign disappears
$3x \div 3 = 36 \div 3$	what we change on the left of the equal sign must also be done on the right
$x = 36 \div 3$	and now the multiplication with 3 has been eliminated
$x = 12$	

In the case of combinations between multiplications/divisions and additions/subtractions, we need to work opposite the BODMAS order to eliminate the additions/subtractions first, and only then the multiplications/subtractions.

$24 + 4y – 36 = 100$	
$24 + 4y – 36 – 24 = 100 – 24$	getting rid of the 24
$4y – 36 = 76$	
$4y – 36 + 36 = 76 + 36$	getting rid of the – 36
$4y = 112$	
$4y \div 4 = 112 \div 4$	getting rid of the multiplication by 4
$y = 112 \div 4 = 28$	

AND FINALLY IN ALL GRADES: TONS OF PRACTICE

Building up a thorough understanding of numbers and operations can really be achieved with the large majority of the children in our primary schools. Only a very minor proportion of our children has serious problems with learning mathematical concepts (such as the numerical equivalent of dyslexia, referred to as dyscalculia). Some children may be a bit slow, but with sufficient practising they can be helped. And practice is what ALL children need in mathematics.

Of course, practice on its own does NOT teach the children maths. The process of teaching and learning needs to spend sufficient time during which the children are to grasp the concepts that they will be working with. Practice before the concept has had a chance to be understood on is a waste of time.

But once children have learnt about mathematical concepts, they need to continue practising their skills. Children do not learn calculating numbers through Maths Main Lessons alone. Several times a week, preferably daily, they need to practise their maths skills.

Areas needing special attention, mostly aiming at Level 3 – instant responses (page 14) are:

- the multiples, followed by the multiplication tables.
- all the bonds up to 20
- tables x10 and x100, e.g. 10 x 5, 20 x 5, 30 x 5 and 1 x 60, 2 x 60, 3 x 60 and 100 x 7, 200 x 7
- adding and subtracting multiples of 10, e.g. 13 + 20, 13 + 30, 13 + 40 and 86 – 10, 86 – 20
- the ability to perform two simple operations, e.g. 1 x 2 + 5, 2 x 2 + 5, 3 x 2 + 5

In written work, the children should get used to managing quite big numbers of calculations, so instead of 5 to 10 sums, make it 20 to 30 sums. Let it be a variety of questions around a common theme, so that children see the cohesion between various presentations of similar problems. Let them check their own work or swap books so that they check each other's work. Incorrect answers are unobtrusively but clearly marked as needing a second try. From time to time the children need to write a standard test to give the teacher a clear picture of what they are able to do in a controlled situation. The teacher should analyse the children's responses to the questions to see how well each different concept has been understood in the class.

Children with backlogs in Maths usually just need to get back to the basics again. There is no way to help a child while trying to skip a level that has not been thoroughly understood. Intensive work, in which the parents of the child will probably need to be involved, will need to be set, with special, attractive exercises (such as working with a jar full of dice) to keep the child motivated. This approach can reassure the child that despite disappointments in Maths, at the level of the basics there is in fact a manageable level, that offers the child reassurance, and the trust that from that level his/her skills can be extended again, with further practice.

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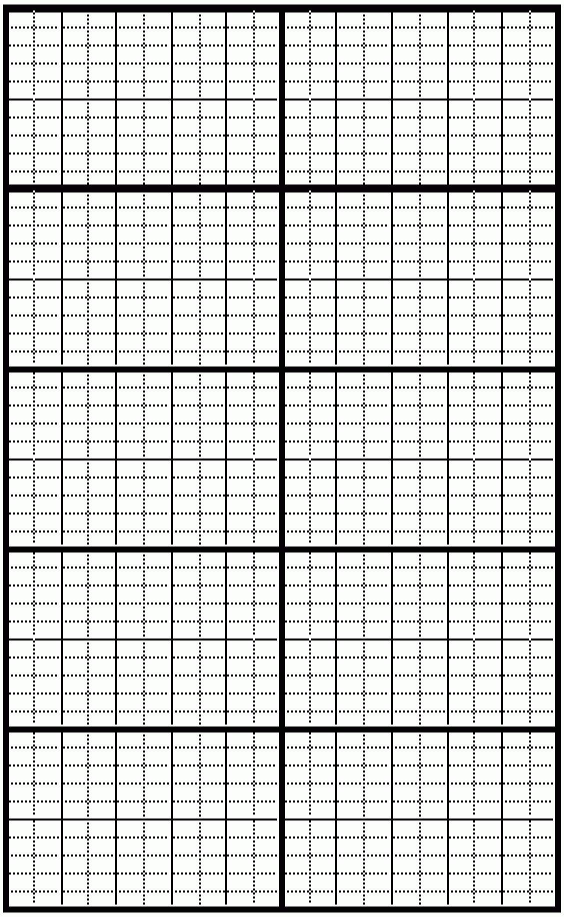
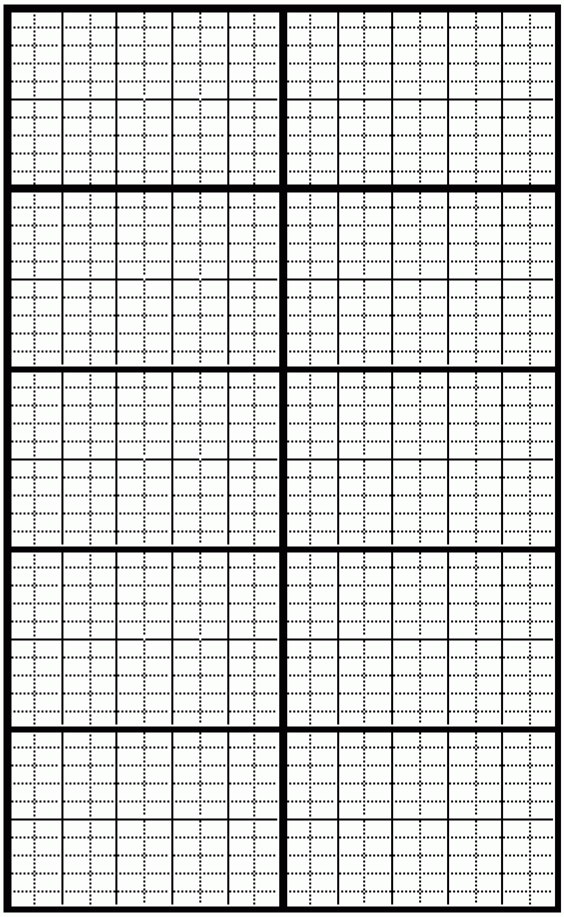
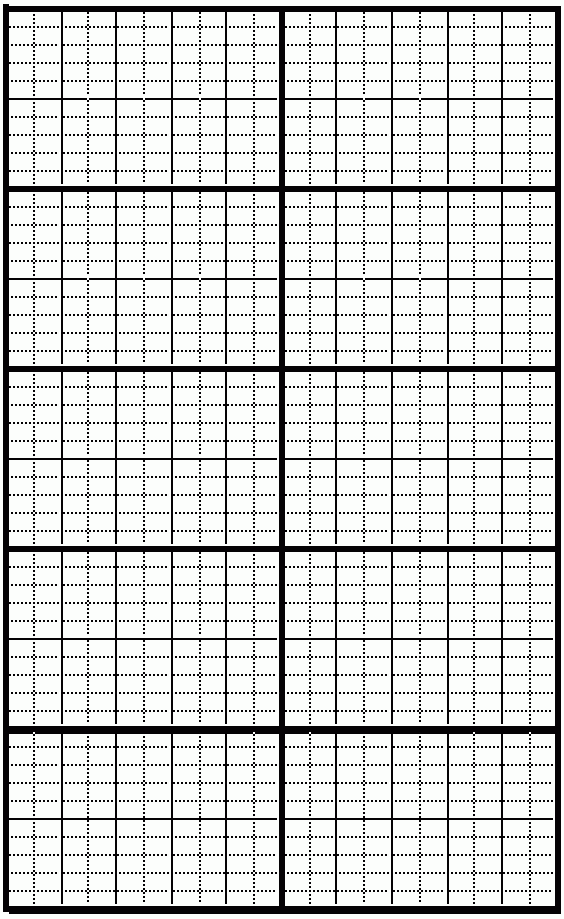
APPENDIX

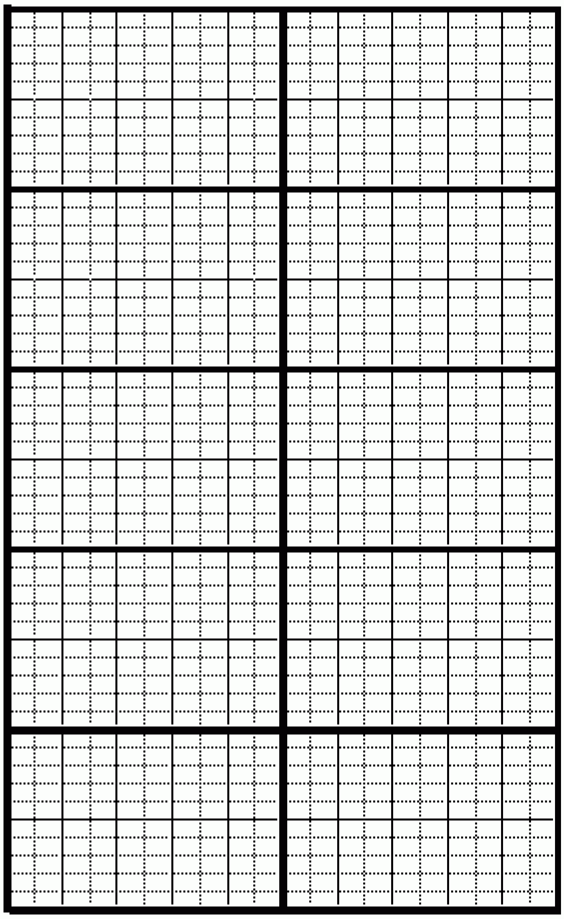
Decimal fractions colour grids from the next page

DECIMAL FRACTION GRIDS (4x)









DECIMAL FRACTION GRID